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ON SMOOTH SOLUTIONS OF NON LINEAR DYNAMICAL SYSTEMS, $f_{n+1} = u(f_n)$, PART I

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ABSTRACT

We consider the dynamical system, $f_{n+1} = u(f_n)$, (1) (where usually n, is time) defined by a continuous map u. Our target is to find a flow of the system for each initial state f_0, i.e., we seek continuous solutions of (1), with the same smoothness degree as u. We start with the introduction of continued forms which are a generalization of continued fractions. With the use of continued forms and a modulator function (i.e., weight function) m, we construct a sequence of smooth functions, which come arbitrarily close to a smooth flow of (1). The limit of this sequence is a functional transform, $\mathcal{K}_m[u]$, of u, with respect to m. The functional transform is a solution of (1), in the sense that, $\mathcal{K}_m[u](y+c)$, is a flow of (1) for each translation constant c. Here we present the first part of our work where we consider a subclass of dissipative dynamical systems in the sence that they have wandering sets of positive measure. In particular we consider strictly increasing real univariate maps, u: $D \to D$, $D = (a+\infty)$, where, $a \le 0$, or, $a = -\infty$, with the property, $u(x)-x \ge \varepsilon > 0$, which implies that u, has no real fixed points. We briefly give some mathematical and physical applications and we discuss some open problems. We demonstrate the method on the simple non-linear dynamical system, $f_{n+1} = (f_n)^2+1$.

Keywords: Non Linear Dynamical Systems, Smooth Flows, Functional Transform, Continuous Iterates, Continued Forms, Abel Functional Equation, Iterative Functional Equations, Iterative Roots

1. INTRODUCTION

1.1. Preliminary Definitions

Let Z, be any totally ordered set of integers. We call an increasing or decreasing sequence of consequtive integers from the set Z, an index set. We denote index sets for the rest of this article as, $Z[k,n]$, $Z[k,\pm\infty)$, $Z(\pm\infty,n]$, where, $k,n \in Z$.

We denote open, halfopen and closed real intervals as, $\mathbb{R}(a,b)$, $\mathbb{R}[a,b)$, $\mathbb{R}(a,b]$, $\mathbb{R}[a,b]$, where for closed boundaries, $a,b \in \mathbb{R}$ and for open boundaries, $a,b \in \mathbb{R} \cup \{\pm\infty\}$, unless otherwise noted.

We say that a function is C^k smooth if it has continuous derivatives of kth order.

We denote the non-negative integer iterates of an univariate function, $f: D \to \mathbb{R}$, where, $f(D) \subset D \subset \mathbb{R}$, as:

$$f^{[n]} \equiv \underbrace{f \circ \ldots \circ f}_{n}, n \in \mathbb{N} \setminus 0$$

$$f^{[0]} \equiv I$$

where, I, is the identity function. We adopt the bracket notation for the iteration exponent to avoid any confusion with powers. The bracket notation has been previously used by Walker (1991) and others. If f, is invertible we denote its inverse for simplicity as f^{-}. If additionaly, $D \subset f(D)$, we define the negative integer iterates of f, as:

$$f^{[-n]} \equiv (f^{-})^{[n]}, n \in \mathbb{N}$$

The existence of the integer iterates, $f^{[n]}$, $\forall n \in \mathbb{Z}$, implies, $D = f(D)$. We will use the following known properties of iterates:

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$$f^{[n]} \circ f^{[m]} = f^{[n+m]}$$
$$(f^{[n]})^{[m]} = f^{[nm]}$$

Especially for the successor function, $S(x) = x+1$, we define real continuous 'principal' iterates as:

$$S^{[a]}(x) \equiv x + a, \ \ a \in \mathbb{R}$$

where, the meaning of 'principal' iterates is explained in section 4.5 on homologous and principal functions.

1.2. Introduction

In this article we seek smooth solutions of the dynamical system Equation 1:

$$f_{n+1} = u(f_n) \tag{1}$$

For a class of functions, $u: D \to D, D = \mathbb{R}(a,+\infty)$, where, $a \leq 0$, or, $a = -\infty$, i.e., we seek smooth flows of (1) through an initial value, $f_0 \in D$.

In section 2 we start with the definition of a continued form, which is a compact representation of successive composition of a sequence of functions, $\{u_j\}$:

$$\underset{j\in\mathbb{Z}[k,n]}{\overset{t}{\mathrm{C}}}[u_j(t)] \equiv u_k \circ \ldots \circ u_n$$

Provided that the domains are such that the compositions can be performed (according to the definition in 1.1). It is straightforward to use continued forms for the representation of continued fractions, continued exponentials, nested radicals, iterates of functions etc. and generally in cases of successive composition.

In section 3 we use continued forms to construct a function space:

$$\mathrm{H} = \{h_{k,n}(y+c), k \leq n \in \mathbb{Z} \cup \{\pm\infty\}, c \in C \subset \mathbb{R}\}$$

which contains the functions:

$$h_{k,n}(y,x) = u^{[k]} \circ \underset{j\in\mathbb{Z}[k,n]}{\overset{t}{\mathrm{C}}}[m(y-j)u(t)] \circ (x)$$
$$h_k(y) = \lim_{n\to+\infty} h_{k,n}(y,x)$$
$$h(y) = \lim_{k\to-\infty} h_k(y) = \mathcal{K}_m[u](y)$$

where, $x \in D$, provided that the limit functions exist. The function, $u \in U$, is a continuous function from a class U, defined in section 3 and, $m \in M$, is a weight function, called a modulator function, which belongs to a class M. We call the limit function, $h = \mathcal{K}_m[u]$, the functional transform of u, with respect to a modulator function m.

In theorem T2 we show that, $\forall u \in U, \exists m \in M$, such that the limit functions, $h_k = \lim_{n\to+\infty} h_{k,n}$, exist for every, $k \in \mathbb{Z}$ and moreover are independent of x.

In theorem T3 we show that, $\forall u \in U, \exists m \in M$, such that the functional transform, $h = \mathcal{K}_m[u]$, exists.

In section 4 we consider an arbitrary finite subset, $S = \{f_n\}, n \in \mathbb{N}[0,q]$, of the orbit of the dynamical system (1), with initial value, $f_0 \in D$. For an arbitrary point, f_r fixed $\in S$ and for some fixed modulator function m, we define a sequence of translation constants, $c_k(f_r, f_0) \in \mathbb{R}$, depending on f_r and f_0, such that, $h_k(r + c_k) = f_r, \forall k \in \mathbb{Z}^-$.

In theorem T4 we show that the smooth functions, $h_k(y + c_k)$, become arbitrarily close to S, as, $k \to -\infty$, in the sence that, $\forall \varepsilon > 0, \exists N \in \mathbb{Z}_0^-$, such that, $\forall k \leq N$, we have, $\| h_k(y + c_k) - S \| = \max_{0\leq n\leq q} |h_k(n + c_k) - f_n| < \varepsilon$. We also show that, provided that the functional transform, $\mathcal{K}_m[u]$, exists, the function, $\mathcal{K}_m[u](y + c)$, interpolates the points of S, where the constant, $c = \lim_{k\to-\infty} c_k$, depends only on f_0 and not on the choice of f_r.

In theorem T5 we show that any functional transform, $\mathcal{K}_m[u]$, is a solution of (1) and that, $\mathcal{K}_m[u](y + c)$, interpolates the complete orbit, $O(f_0)$, of the dynamical system (1). Thus, $\mathcal{K}_m[u](y + c)$, is a flow through f_0, of the dynamical system (1).

In this first part of our work we consider a subclass of dissipative dynamical systems in the sence that they have wandering sets of positive measure. In particular we consider strictly increasing real univariate maps, $u: D \to D, D = \mathbb{R}(a,+\infty)$, where, $a \leq 0$, or, $a = -\infty$, with the property, $u(x) - x \geq \varepsilon > 0$, which implies that u, has no real fixed points.

From (Belitskii and Lyubich, 1999) we have that C^k smooth solutions of (1) exist, $\forall k \in \mathbb{N} \cup \{+\infty, \omega\}$, provided that u, is C^k smooth and, $f^{-\prime} \neq 0$. We believe that the functional transform, $\mathcal{K}_m[u]$, delivers indeed these smooth solutions.

For continuous functions u and f, the dynamical system (1) is equivalent with the Abel Functional Equation (AFE), $f(x+1) = u \circ f(x)$. We also state a known lemma which gives a general solution of the AFE from any particular solution.

In section 5 on mathematical and physical applications we briefly discuss the subject of the formal definition of continuous iterates of any function in the class *U*, where we propose a split of iterates into two categories: Principal iterates and homologous iterates. Finally we demonstrate the method on the simple non-linear dynamical system, $f_{n+1} = (f_n)^2 + 1$, defined by the map, $u(x) = x^2 + 1$. We show a smooth solution of the system using the logistic function as a modulator function, along the way pointing out some computational difficulties. At the end we give some important open problems related to this article.

2. CONTINUED FORMS

The terms 'continued form' or 'continued composition' seem not to have been used in mathematics as all encompassing names for successive composition of a sequence of functions. In contrast more restricted terms such as, continued fractions, continued powers, continued roots, continued radicals, continued exponentials etc., invariably indicating successive composition, often appear in use. It seems plausible that they all ultimately rely on 'continued fractions', a name first introduced by John Wallis in his Opera Mathematica in 1695 (Olds 1963), but as a mathematical entity it is known since antiquity. For ex. continued fractions are implied in Euclid's Elements, as a subresult of his algorithm for the greatest common divisor and also used by the Indian mathematician Aryabhata in the 6 th cent., in his solution of indeterminate equations (Olds 1963).

The relatively rare general term' Kettenoperationen' in german has more or less the same meaning as continued forms, but nevertheless seems not to have been used in a general setting. We will now give a formal definition of continued forms (the equivalent in german would be Kettenformen).

Definition

Let, $\{u_j : D_j \to \mathbb{C}\}, D_j \subset \mathbb{C}, j \in \mathbb{Z}[k,n]$, be a sequence of univariate functions such that:

$$\begin{aligned} &u_j(D_j) \subset D_{j-1}, \forall j \in \mathbb{Z}[k+1,n] \quad if, k < n \\ &u_j(D_j) \subset D_{j+1}, \forall j \in \mathbb{Z}[k-1,n] \quad if, k > n \end{aligned}$$

We call the ordered composition of consequtive functions of this sequence a continued composition or a continued form. We denote a continued form in a compact way as:

$$\underset{j\in\mathbb{Z}[k,n]}{\overset{t}{\mathrm{C}}}[u_j(t)] \equiv u_k \circ \ldots \circ u_n$$

where, the dummy variable *t*, is the composition variable. A continued form may be evaluated at a point, $x \in D_n$, in which case we write:

$$\underset{j\in\mathbb{Z}[k,n]}{\overset{t}{\mathrm{C}}}[u_j(t)] \circ (x) \equiv u_k \circ \ldots \circ u_n(x)$$

We call *x*, the starting variable because evaluation begins with *s*. Continued composition of multivariate functions is performed with respect to one particular composition variable, which must be the same throughout the continued form. A continued form has an infinite number of terms if the index set is infinite, as for example in, $\underset{j\in\mathbb{Z}[k,+\infty)}{\overset{t}{\mathrm{C}}}[u_j(t)]$, $\underset{j\in\mathbb{Z}}{\overset{t}{\mathrm{C}}}[u_j(t)]$, etc. In this case the continued form notation represents a formal expression and the limits may not exist. To avoid some parenthesis we define that inside an expression, a continued form has precedence over composition of functions and over binary abelian operations such as addition and multiplication.

A continued form is a function since it represents a composition of functions. Nevertheless an infinite continued form may not converge towards any particular function, either pointwise or uniformly.

Definition

Let, $CF = \underset{j\in\mathbb{Z}[k,+\infty)}{\overset{t}{\mathrm{C}}}[u_j(t)]$, be an infinite continued form. We consider the partial continued forms:

$$g_n = \underset{j\in\mathbb{Z}[k,n]}{\overset{t}{\mathrm{C}}}[u_j(t)], \text{ where, } k \le n$$

We say that *CF*, converges pointwise or uniformly to a function *g*, if and only if, the sequence of functions, $\{g_n\}$, converges pointwise or uniformly to *g*, as, $n \to +\infty$, respectively. Analogous for:

$$CF = \underset{j\in\mathbb{Z}(-\infty,n]}{\overset{t}{\mathrm{C}}}[u_j(t)].$$

In the case of a biinfinite continued form the two limits are independent unless otherwise noticed.

The most important property of continued forms is the following, which stems from the associative property of composition.

Corollary

Let, $s_{k,n}(x) = \mathop{\mathrm{C}}_{j\in\mathbb{Z}[k,n]}^{t} [u_j(t)]\circ(x)$, be a continued form and let, $l\in\mathbb{Z}[k,n]$. Then:

$$s_{k,n}(x) = s_{k,l}\circ s_{l,n}(x)$$

Mathematical expressions involving repeated composition may be nicely represented as continued forms. Some examples are:

Sums:

$$\sum_{j\in\mathbb{Z}[k,n]} a_j = \mathop{\mathrm{C}}_{j\in\mathbb{Z}[k,n]}^{t}[a_j+t]\circ(0)$$

Products:

$$\prod_{j\in\mathbb{Z}[k,n]} a_j = \mathop{\mathrm{C}}_{j\in\mathbb{Z}[k,n]}^{t}[a_j t]\circ(1)$$

Taylor series:

$$f(x) = \mathop{\mathrm{C}}_{j\in\mathbb{N}[0,n]}^{t}\left[f^{(j)}(a)+\frac{x-a}{j+1}t\right]\circ(f^{(n+1)}(\xi))$$

where, $\xi\in\mathbb{R}[a,x]$ (Lagrange form)

Iterates of functions:

$$u^{[n]}(x) = \mathop{\mathrm{C}}_{j\in\mathbb{N}[1,n]}^{t}[u(t)]\circ(x)$$

Continued fractions:

$$a_0+\cfrac{b_1}{a_1+\cfrac{b_2}{a_2+\ldots\cfrac{b_n}{a_n}}} = a_0+\mathop{\mathrm{C}}_{j\in\mathbb{N}[1,n]}^{t}\left[\frac{b_j}{a_j+t}\right]\circ(0)$$

Continued roots (or radicals):

$$\sqrt{a_1+b_1\sqrt{a_2+\ldots+b_{n-1}\sqrt{a_n+b_n}}} = \mathop{\mathrm{C}}_{j\in\mathbb{N}[1,n]}^{t}\left[\sqrt{a_j+b_j t}\right]\circ(1)$$

Continued powers:

$$(a_1+(a_2+\ldots+(a_n)^2\ldots)^2)^2 = \mathop{\mathrm{C}}_{j\in\mathbb{N}[1,n]}^{t}\left[(a_j+t)^2\right]\circ(0)$$

$$a_1+(a_2+(a_3+\ldots+(a_n)^n\ldots)^3)^2 = \mathop{\mathrm{C}}_{j\in\mathbb{N}[1,n]}^{t}\left[(a_j+t)^j\right]\circ(0)$$

Continued exponentials (towers):

$$(a_0)^{(a_1)^{(a_2)^{\cdot^{\cdot^{(a_n)}}}}} = \mathop{\mathrm{C}}_{j\in\mathbb{N}[0,n]}^{t}\left[(a_j)^t\right]\circ(1)$$

The value of continued forms generally depends on the starting variable. Nevertheless the value of converging infinite continued forms may not depend on the starting variable. An example are taylor series evaluated within the radius of convergence:

$$f(x) = \lim_{n\to+\infty}\mathop{\mathrm{C}}_{j\in\mathbb{N}[0,n]}^{t}\left[f^{(j)}(a)+\frac{x-a}{j+1}t\right]\circ(f^{(n+1)}(\xi)) = \mathop{\mathrm{C}}_{j\in\mathbb{N}}^{t}\left[f^{(j)}(a)+\frac{x-a}{j+1}t\right]$$

For continued powers see Bo-Yan and Feng (2013) and Jones (1995).

3. THE FUNCTION SPACE H

3.1. Definitions

We now formally define the function space H, which depends on a class of continuous functions *U* and on a class of modulator functions *M*. In the next sections we will show that the limits of particular sequences of functions from H, are solutions of (1).

Definition

Let *U*, *M*, be classes of continuous functions. We define a function space H, as:

$$\mathrm{H} = \{h_{k,n}(y+c),\ k\le n\in\mathbb{Z}\cup\{\pm\infty\},\ c\in C\subset\mathbb{C}\}$$

where, $h_{k,n}$, are functions defined as:

$$h_{k,n}(y,x) = u^{[k]} \circ \mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t} [m(y-j)u(t)] \circ (x)$$
$$h_{k,+\infty}(y,x) = \lim_{n\to+\infty} h_{k,n}(y,x)$$
$$h_k(y) = h_{k,+\infty}(y,x)$$
$$h(y) = \lim_{k\to-\infty} h_k(y) = \mathcal{K}_m[u](y)$$

where, x, is in the domain of, $u \in U$, $m \in M$ and, $k \le n \in \mathbb{Z}$. We call, $\mathcal{K}_m[u]$, the functional transform of u, with respect to m. The function classes U, M, will be defined such that, $\exists y_1 \in \mathbb{R} \cup \{-\infty\}$, such that, $\forall y \in \mathbb{R}(y_1,+\infty)$, the following existence conditions are satisfied:

C1: The functions, $h_{k,n}(y,x)$, exist, $\forall(u,m)\in U\times M$, $\forall k \le n \in \mathbb{Z}$, for every x, in the domain of u

C2: The limit functions, $h_{k,+\infty}(y,x)$, exist, $\forall(u,m)\in U\times M$, $\forall k\in\mathbb{Z}$

C3: $\forall u\in U$, $\exists m\in M$, such that the limit functions, $h_k(y)$, are independent of the starting variable x, $\forall k\in\mathbb{Z}$

C4: $\forall u\in U$, $\exists m\in M$, such that the functional transform, $h(y)=\mathcal{K}_m[u](y)$, exists

The conditions C1-C4 pose restrictions to the function classes U and M. The following definitions of U, M, are chosen, such that these conditions are satisfied.

Definition

We define a class U, of continuous functions with the properties:

- $u: D \to D,\ D = \mathbb{R}(a,+\infty)$, where, $a \le 0$, or, $a = -\infty$
- u, is strictly increasing
- $u(x) - x \ge \varepsilon > 0$

Definition

We define a class M, of C^k smooth functions with the properties:

- $m: \mathbb{R} \to \mathbb{R}(0,1)$
- $\lim_{x\to-\infty} m(x) = 0 \wedge \lim_{x\to+\infty} m(x) = 1$

We call the functions in M, modulator functions.

In the following existence theorems we show that the conditions required by the definition of H, are satisfied by the functions in U, M.

3.2. Existence Theorems

The following lemma satisfies the existence condition C1 of the definition of H.

Lemma 1: Assumptions

A1. $u\in U$
A2. $m\in M$

Propositions

P1. For every, $q\in\mathbb{Z}$ we have:

$$\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t} [m(y-j)u(t)] \circ (x) = \mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k+q,n+q]}^{t} [m(y+q-j)u(t)] \circ (x)$$

P2. $\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t} [m(y-j)u(t)] \circ (x)$, is strictly increasing with respect to x, $\forall(u,m)\in U\times M$, $\forall k\le n\in\mathbb{Z}$, $\forall y$ fixed$\in\mathbb{R}$ (assuming its existence)

P3. The continued form, $\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t} [m(y-j)u(t)] \circ (x)$, exists, $\forall(u,m)\in U\times M$, $\forall k\le n\in\mathbb{Z}$, $\forall y\in\mathbb{R}$ and, $\forall x\in D$

P4. $\exists y_1\in\mathbb{R}\cup\{-\infty\}$, such that the functions:

$$h_{k,n}(y,x) = u^{[k]} \circ \mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t} [m(y-j)u(t)] \circ (x)$$

Exist, $\forall(u,m)\in U\times M$, $\forall k\le n\in\mathbb{Z}$, $\forall y\in\mathbb{R}(y_1,+\infty)$ and, $\forall x\in D$.

This lemma satisfies condition C1 of the definition of H.

Proof: P1:

$$\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t} [m(y-j)u(t)] \circ (x) =$$
$$\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t} [m(y+q-(j+q))u(t)] \circ (x) =$$
$$\mathop{\mathrm{C}}\limits_{(j-q)\in\mathbb{Z}[k,n]}^{t} [m(y+q-j)u(t)] \circ (x) =$$
$$\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k+q,n+q]}^{t} [m(y+q-j)u(t)] \circ (x)$$

P2: Since u, is strictly increasing, $m(y-n)u(x)$, is strictly increasing with respect to x, for fixed y and, $\forall n \text{ fixed} \in \mathbb{Z}$. Then, $\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t}[m(y-j)u(t)]\circ(x)$, is strictly increasing with respect to x, since it is a composition of strictly increasing functions.

P3: For, $y \in \square$, let, $b_{k,n}(y) = \lim\limits_{x\to a^+}\left(\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t}[m(y-j)u(t)]\circ(x)\right)$, (for a, see definition of U). Then for, $(u,m)\in U\times M$ and, $n\in\mathbb{Z}$, we will show by induction that: $b_{k,n}(y) > a, \forall k \le n$

- $b_{n,n}(y) = \lim\limits_{x\to a^+}\left(\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[n,n]}^{t}[m(y-j)u(t)]\circ(x)\right)$
  $= \lim\limits_{x\to a^+}\left(m(y-n)u(x)\right) = m(y-n)\lim\limits_{x\to a^+}u(x) > a$,
  since, $\lim\limits_{x\to a^+}u(x) > a \le 0$ and, $0 < m(y-n) < 1$
- We assume that, $b_{n-N,n}(y) > a$, where, $N \in \square \setminus 0$
- $b_{n-(N+1),n}(y) =$
  $\lim\limits_{x\to a^+}\left(\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[n-(N+1),n]}^{t}[m(y-j)u(t)]\circ(x)\right) =$
  $\lim\limits_{x\to a^+} m(y-(N+1))u\circ\left(\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[n-N,n]}^{t}[m(y-j)u(t)]\circ(x)\right)$
  $= m(y-(N+1))u(b_{n-N,n}(y)) > a$, since,
  $u(b_{n-N,n}(y)) \ge b_{n-N,n}(y) > a \le 0$, and,
  $0 < m(y-(N+1)) < 1$

Thus, $b_{k,n}(y) > a, \forall k \le n$. Since, $\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t}[m(y-j)u(t)]\circ(x)$, is strictly increasing with respect to x, then, $\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t}[m(y-j)u(t)]\circ(D)\in D$ and thus the continued forms exist, $\forall k \le n \in \mathbb{Z}$.

P4a: The iterate, $u^{[n]}$, exists, $\forall n \in \square$, since, $u(D)\subset D$. Let, $a_n = \lim\limits_{x\to a^+}u^{[n]}(x), n\in\square$, then since u, is strictly increasing we have that, $u^{[n]}$, is strictly increasing thus, $u^{[n]}(D) = \square(a_n,+\infty)$. Since, $u(x) > x$, we have, $a_{n+1}\ge a_n \ge a$ and, $\begin{matrix} a_n \to +\infty & \text{if}, a_1 > a \\ a_n = a & \text{if}, a_1 = a \end{matrix}$. We are interested in the images of the negative iterates, $u^{[k]}$, which thus have the property, $\forall k \ge -n, u^{[k]}(a_n) \ge a$ and since, $u^{[k]}$, are strictly increasing then, $u^{[k]}(\square(a_n,+\infty)) \in D$

P4b: Let, $s_{k,n}(y) = \mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t}[m(y-j)u(t)]\circ(x)$, where, $y\in\square$. We will show that, $\forall x \text{ fixed} \in D$, $\forall n \text{ fixed} \in \mathbb{Z}$, we have: $\lim\limits_{k\to-\infty}s_{k,n}(y) = +\infty$. By the ratio test we have: $\lim\limits_{k\to-\infty}\dfrac{s_{k,n}(y)}{s_{k+1,n}(y)}$

$$= \lim_{k\to-\infty}\frac{m(y-k)u\circ s_{k+1,n}(y)}{s_{k+1,n}(y)}$$
$$= \lim_{k\to-\infty}m(y-k)\lim_{k\to-\infty}\frac{u\circ s_{k+1,n}(y)}{s_{k+1,n}(y)}$$
$$= \lim_{k\to-\infty}\frac{u\circ s_{k+1,n}(y)}{s_{k+1,n}(y)} > 1$$

true since, $\lim\limits_{k\to-\infty}m(y-k) = 1$ and, $u(x) - x \ge \varepsilon > 0 \overset{x\neq 0}{\Rightarrow} \dfrac{u(x)}{x} > 1+\varepsilon > 1$.

Thus for, $y=0$ and arbitrary x, n, we have, $\lim\limits_{k\to-\infty}s_{k,n}(0) = +\infty$, hence, $\forall r\in\square$, there is some, $L\le 0 \wedge L\le n$, such that, $s_{l,n}(0) > a_r, \forall l \le L$. Then from P4a we have, $u^{[k]}\left(s_{l,n}(0)\right) > u^{[k]}(a_r) > a, \forall k \ge -r$. Then if, $k \le L$, we have, $u^{[k]}\left(s_{k,n}(0)\right) > a$ and, $h_{k,n}(0,x) = u^{[k]}\circ\mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t}[m(-j)u(t)]\circ(x)$, exists, $\forall k \ge -r$. If, $k > L$, we set, $k - L = q \in \square$. Then from P1:

$$u^{[k]}\left(s_{L,n}(0)\right) = u^{[k]}\circ\mathop{\mathrm{C}}_{j\in\mathbb{Z}[L,n]}^{t}[m(-j)u(t)]\circ(x) = u^{[k]}\circ$$
$$\mathop{\mathrm{C}}_{j\in\mathbb{Z}[L+q,n+q]}^{t}[m(q-j)u(t)]\circ(x) = u^{[k]}\circ\mathop{\mathrm{C}}_{j\in\mathbb{Z}[k,n+q]}^{t}$$
$$[m(q-j)u(t)]\circ(x)$$

Thus, $h_{k,n+q}(q,x)$, exists, $\forall k \ge -r$ and since n, is arbitrary we conclude that, $h_{k,n}(q,x)$, exists, $\forall k \ge -r$. From P1 we have that, $h_{k,n}(y,x)$, exists, $\forall y \ge q$. But q, depends only on L, for fixed k and since L, always exists we have that q, always exists for every, $k \le n \in \mathbb{Z}$. Thus condition C1 of the definition of H, is satisfied. This concludes the proof.

The following theorem satisfies the existence conditions C2 and C3 of the definition of H.

Theorem 2: Assumptions

A1. $u \in U$
A2. $m \in M$
A3. $h_{k,n}(y,x) \in \mathrm{H}$, satisfy condition C1 of the definition of H.
A4. $y_1 \in \mathbb{R} \cup \{-\infty\}$

Propositions

P1. The limit functions, $h_{k,+\infty}(y) = \lim\limits_{n\to+\infty} h_{k,n}(y,x)$, exist, $\forall (u,m) \in U \times M$, $\forall k \in \mathbb{Z}$, $\forall y \in \mathbb{R}(y_1,+\infty)$. This satisfies condition C2 of the definition of H.

P2. $\forall u \in U$, $\exists m \in M$, such that the limit functions, $h_k(y) = h_{k,+\infty}(y,x)$, are independent of the starting variable x, $\forall k \in \mathbb{Z}$, $\forall y \in \mathbb{R}(y_1,+\infty)$. This satisfies condition C3 of the definition of H.

Proof

P1. We will first prove this proposition for, $h_0(y) = h_{0,+\infty}(y,x)$. For fixed, $y \in \mathbb{R}$ and, $m \in M$, we define the sequence: $c_n = m(y-n), n \in \mathbb{N}$.

Let, $s_n(x) = \underset{j\in\mathbb{Z}[0,n]}{\overset{t}{\mathrm{C}}} [c_j u(t)] \circ (x)$. From L1 the continued forms exist and are strictly increasing with respect to, $x \in D$, $\forall (u,m) \in U \times M$, $\forall n \in \mathbb{Z}$.

Let, $s(x) = \lim\limits_{n\to+\infty} s_n(x)$. We will show that, $s(x) \in \mathbb{R}$, $\forall u \in \mathbb{U}$, $\forall x \in D$.

A: Let, x fixed >0, then:

- $c_n u(x) > 0, \forall n \in \mathbb{N}$
  (since, $u(0), c_n > 0$)
- $c_n u(x)$, is strictly increasing, $\forall n \in \mathbb{N}$
  (since, $c_n > 0$ and u, is strictly increasing)
- $\exists N \in \mathbb{N}$, such that, $\forall n \geq N$, we have, $c_n u(x) < x \Rightarrow$
  (since, $c_n \to 0$ and, $x, u(x) > 0$)
- $0 < c_n u(0) < c_n u(x) < x \Rightarrow$
  (since, x>0 and u, is strictly increasing)
- $c_n u(\mathbb{R}[0,x]) \subset \mathbb{R}[0,x] \Rightarrow$
- $s_n(\mathbb{R}[0,x]) \subset \mathbb{R}[0,x]$
- $s_n(0) = s_{n-1}(c_n u(0)) > s_{n-1}(0), \forall n \geq N \Rightarrow$
  (since, $c_n u(0) > 0$ and, $s_n(x)$, is strictly increasing with respect to x)
- $s_n(0) > s_0(0) = c_0 u(0) > 0 \Rightarrow$
- $\{s_n(0)\}_{n\geq N}$, is a strictly increasing sequence
- $s_n(x) = s_{n-1}(c_n u(x)) < s_{n-1}(x), \forall n \geq N \Rightarrow$
  (since, $c_n u(x) < x$ and, $s_n(x)$, is strictly increasing with respect to x)
- $\{s_n(x)\}_{n\geq N}$, is a strictly decreasing sequence
- $0 < s_n(0) < s_n(x) < x, \forall n \geq N \Rightarrow$
- the limits, $s(0), s(x)$, exist and moreover, $0 < s(0) < s(x) < x$
  This proves P1 for, $k = 0$ and, $x > 0$.

B: If, x = 0, let, x_1 = $c_n u(0)$>0 and we are done by A. Otherwise let fixed x, be such that, a<x<0. Then:

a. If, $\lim\limits_{x\to a^+} u(x) \geq 0$,we let, $x_1 = c_n u(x) > 0, \forall x \in \mathbb{R}(a,0)$ and we are done by A
b. If, $\lim\limits_{x\to a^+} u(x) < 0$, there exists, $b < 0$, such that, $u(b) = 0$. Then:

i. Let, $x \in \mathbb{R}[b,0)$, then let, $x_1 = c_n u(x) \geq 0$ and we are done by A
ii. Let, $a < x < b < 0$, which implies that, $u(x) < 0$, then, $\exists N \in \mathbb{N}$, such that, $\forall n \geq N$, we have, $c_n u(x) > b$, (since, $c_n \to 0$ and, b, $u(x) < 0$. We let, $x_1 = c_n u(x)$ and we are done by Bbi

Thus, $s(x) = \lim\limits_{n\to+\infty} \underset{j\in\mathbb{Z}[0,n]}{\overset{t}{\mathrm{C}}} [c_j u(t)] \circ (x)$, exists, $\forall u \in \mathbb{U}$ and, $\forall x \in D$. Since the sequence, $\{c_n\}$, is arbitrary and exists, $\forall m \in M$, we have that:

$$h_{0,+\infty}(y) = \lim_{n\to+\infty} h_{0,n}(y,x) = \lim_{n\to+\infty} \underset{j\in\mathbb{Z}[0,n]}{\overset{t}{\mathrm{C}}} [m(y-j)u(t)] \circ (x),$$
$$\text{exists}, \forall (u,m) \in U \times M, \forall x \in D, \forall y \in \mathbb{R}$$

Then by L1, $h_{k,+\infty}(y) = \lim\limits_{n\to+\infty} h_{k,n}(y,x) = \lim\limits_{n\to+\infty} \underset{j\in\mathbb{Z}[k,n]}{\overset{t}{\mathrm{C}}} [m(y-j)u(t)] \circ (x)$, exists, $\forall (u,m) \in U \times M$, $\forall x \in D$, $\forall y \in \mathbb{R}(y_1,+\infty)$.

This completes the proof of P1:

P2. Let, $\delta_n(x) = s_n(x) - s_n(0), x \in D$. We will prove that, $\delta_n(x) \to \delta(x) = 0$, as, $n \to +\infty$. If, $x < 0$, we are done. In P1 we have shown that all cases where, $x < 0$, can be reduced to, $x < 0$, where in this case we have:

- $0 < s_n(0) < s_{n+1}(0) < s_{n+1}(x) < s_n(x) \Rightarrow$

(from P1A)

- $0 < s_{n+1}(x) - s_{n+1}(0) < s_n(x) - s_n(0) \Rightarrow$
- $0 < \delta_{n+1}(x) < \delta_n(x) \Rightarrow \dfrac{\delta_{n+1}(x)}{\delta_n(x)} = \dfrac{s_{n+1}(x) - s_{n+1}(0)}{s_n(x) - s_n(0)} =$

  $\dfrac{s_n(c_{n+1}u(x)) - s_n(c_{n+1}u(0))}{s_n(x) - s_n(0)} < 1, \forall n \ge N \Rightarrow$
- $\delta_n(x) \to \delta(x) \in \square$, as, $n \to +\infty$

This condition is not sufficient to show that, $\delta(x) = 0$, for every sequence, $\{c_n\}_{n\in\square}$. Nevetheless since, $c_n u(x) \to 0$ and, $c_n u(0) \to 0$, there is always a sequence, $\{c_n\}_{n\in\square}$, converging to zero fast enough such that, $\forall n \ge N$, $\exists 0<\varepsilon<1$, such that, $s_n(c_{n+1}u(x)) - s_n(c_{n+1}u(0)) \le \varepsilon(s_n(x) - s_n(0)) \Rightarrow \dfrac{\delta_{n+1}(x)}{\delta_n(x)} \le \varepsilon < 1$ and this finally implies that, $\delta(x) = 0$, $\forall x \in D$.

Combining this result with the existence of, $s(x)$, $\forall x \in D$, from P1, we conclude that the limit, $s(x)$, is independent of x. Applying the same arguments as at the end of P1B above, we conclude that, $\forall u \in U$, $\exists m \in M$, such that the limit functions, $h_k(y) = h_{k,+\infty}(y,x)$, are independent of the starting variable x, $\forall k \in \mathbb{Z}$, $\forall y \in \square(y_1, +\infty)$ and this satisfies condition C3 of the definition of H.

The following theorem satisfies the existence condition C4 of H.

Theorem 3: Assumptions

A1. $u \in U$
A2. $m \in M$
A3. $h_{k,n}(y,x) \in \mathrm{H}$, satisfy condition C1 of the definition of H.
A4. $h_k(y) \in \mathrm{H}$, satisfy condition C2, C3 of the definition of H
A5. $y_1 \in \square \cup \{-\infty\}$

Propositions

P1. $\forall u \in U$, $\exists m \in M$, such that the functional transform, $h(y) = \mathcal{K}_m[u](y)$, exists, $\forall y \in \square(y_1, +\infty)$.

Proof

P1. For, m fixed $\in M$, we define the sequence: $d_k = m(y-k), k \in \mathbb{Z}^-$. Let, $x \in D$ and let:

$$r_k(x) = \underset{j\in\mathbb{Z}[k,-1]}{\overset{t}{\mathrm{C}}} [d_j u(t)] \circ (x) \text{ and, } p_k(x) = u^{[k]} \circ r_k(x)$$

By A3 both r_k and p_k , exist, $\forall k \in \mathbb{Z}^-$ and $\forall y \in \square(y_1, +\infty)$. We want to show that, $\forall u \in U$, $\exists m \in M$, such that, $\lim\limits_{k\to-\infty} p_k(x)$, exists, $\forall x \in D$ and $\forall y \in \square(y_1, +\infty)$.

To simplify this proof we set, $n = -k$, $c_n = m(y+n)$, $s_n(x) = \underset{j\in\square[n,1]}{\overset{t}{\mathrm{C}}} [c_j u(t)] \circ (x)$ and, $q_n(x) = u^{[-n]} \circ s_n(x)$, where, $n \in \square$. Then:

- $u^{[n]}(x)$, is strictly increasing, $\forall n \in \mathbb{Z}$
  (iterate and inverse of a strictly increasing function)
- $c_n u(x)$, is strictly increasing, $\forall n \in \square$
  (since, $c_n > 0$)
- $s_n(x)$, is strictly increasing, $\forall n \in \square \setminus 0$
  (a composition of strictly increasing functions)
- $\exists x_1 > 0 : s_1(x) = c_1 u(x) = u(x - x_1)$
  (since, $c_n < 1$)
- $\exists x_2 > 0 : s_2(x) = c_2 u(c_1 u(x)) =$
- $c_2 u \circ u(x - x_1) = u^{[2]}(x - x_1 - x_2)$
- $\exists x_j > 0, j \in \square[1,n]:$
- $s_n(x) = c_n u^{[n]}\left(x - \sum\limits_{1\le j\le n-1} x_j\right) = u^{[n]}\left(x - \sum\limits_{1\le j\le n} x_j\right) \Rightarrow$
- $q_n(x) = u^{[-n]} \circ s_n(x) = u^{[-n]} \circ u^{[n]}\left(x - \sum\limits_{1\le j\le n} x_j\right) \Rightarrow$

  $q_n(x) = x - \sum\limits_{1\le j\le n} x_j$

  (this is easily proved by induction)

Thus, $q_n(x)$, converges, if and only if, the series, $\sum\limits_{1\le j\le n} x_j$, converges. We have:

$$\lim_{n\to+\infty} q_n(x) = \lim_{n\to+\infty}\left(x - \sum_{1\le j\le n} x_j\right) = x - \sum_{1\le j} x_j$$

The series, $\sum\limits_{1\le j} x_j$, converges if and only if (since, x_n>0):

$$\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} < 1 \Leftrightarrow \lim_{n\to+\infty} \frac{x_{n+1}}{x_n} = 1 - \delta$$

where, $\delta > 0$, (independent of n). We have:

$$x_n = q_{n-1}(x) - q_n(x) = q_{n-1}(x) - u^{[-n]} \circ s_n(x) \Rightarrow$$
$$x_n = q_{n-1}(x) - u^{[-n]} \circ c_n u^{[n]} \circ q_{n-1}(x)$$

For, $n\,\text{fixed} \in \mathbb{N} \setminus 0$, $x\,\text{fixed} \in D$ and fixed, c_j, we have fixed, $x_j > 0$, where, $j = 1, 2, \ldots n-1$. This implies that, x_n, depends only on, c_n. Let assume that, c_n, can take any value, $0 < c_n < 1$.

Let, $x_n(c_n) = q_{n-1}(x) - u^{[-n]} \circ c_n u^{[n]} \circ q_{n-1}(x)$, then for, $c_n = 1 \Rightarrow x_n(1) = q_{n-1}(x) - u^{[-n]} \circ u^{[n]} \circ q_{n-1}(x) = 0$ and as c_n, decreases from 1, x_n, increases (since the function, $x_n(c_n)$, is strictly decreasing with respect to c_n). Thus (since x_n, varies continuously with continuous c_n) for any, $\delta > 0, \exists \varepsilon_n > 0$, such that: $1 - \varepsilon_n < c_n < 1 \Rightarrow 0 < x_n < x_{n-1} + \delta \Rightarrow$ $\exists \varepsilon_{n+1} > 0$, such that, $1 - \varepsilon_{n+1} < c_{n+1} < 1 \Rightarrow \frac{x_{n+1}}{x_n + \delta} < 1$ (since n, is arbitrary and δ, does not depend on n), $\Rightarrow \frac{x_{n+1}}{x_n} \le \delta_1 < 1$ (where, $\delta_1 > 0$, does not depend on n), $\Rightarrow \lim_{n \to +\infty} \frac{x_{n+1}}{x_n} < 1$.

Thus the series, $\sum_{1 \le j} x_j$, converges if and only if, $\exists N \in \mathbb{N} \setminus 0$, such that, $1 - \varepsilon_n < c_n < 1, \forall n \ge N$.

It is easy to show that, $\varepsilon_n \to 0, \text{as}, n \to +\infty$, thus in other words the series converges if c_n, tends to 1, sufficiently fast. Notice that the rate of convergence depends on the choice of m. It can also be shown that for fast increasing functions u, the rate of convergence of c_n, may be slow and vice versa.

Thus, $\forall u \in U$, $\exists m \in M$, such that, $\lim_{n \to +\infty} q_n(x)$, exists, $\forall x \in D$, which implies that, $\forall u \in U$, $\exists m \in M$, such that, $\lim_{k \to -\infty} p_k(x)$, exists, $\forall x \in D$ and, $\forall y \in \mathbb{R}(y_1, +\infty)$.

By A4, $h_0(y)$, exists according to C3. Setting, $x = h_0(y)$, we have that:

$$h(y) = \mathcal{K}_m[u](y) = \lim_{k \to -\infty} p_k \circ h_0(y)$$

Exists as required by condition C4 of the definition of H.

4. SOLUTIONS OF THE DYNAMICAL SYSTEM, $f_{n+1} = u(f_n)$

4.1. Smooth Approximations of the Solutions

We consider an arbitrary finite subset, $S = \{f_n\}, 0 \le n \le q \in \mathbb{N}$, of the orbit of, $f_{n+1} = u(f_n)$ (1), depending on an initial value f_0, in the domain of u. For some fixed modulator function, $m \in M$, such that, $h_k(y)$, exists, $\forall y \in \mathbb{R}(y_1, +\infty)$, we consider an arbitrary point, $f_r\,\text{fixed} \in S$. We define a sequence of translation constants, $c_k(f_r, f_0) \in \mathbb{R}$, depending on f_r and f_0, such that, $h_k(r + c_k) = f_r, \forall k \in \mathbb{Z}^-$. The translation constants will necessarily be such that, $r + c_k \ge y_1$.

In the next theorem we show among others that the smooth functions, $h_k(y + c_k)$, become arbitrarily close to all points of S, as, $k \to -\infty$.

Theorem 4: Assumptions:

A1. Let, $f_{n+1} = u(f_n), n \in \mathbb{N}$

A2. $u \in U$ and, $m \in M$

A3. $h_k(y), h(y) \in \mathrm{H}$

A4. Let, $h_0(y), u, m$, be C^p smooth functions where, $p \in \mathbb{N} \cup \{+\infty, \omega\}$, where by C^ω smooth we denote real analytic functions

A5. Let, $S = \{f_n\}, n \in \mathbb{N}[0, q]$

A6. Let, $\{c_k\}_{k \in \mathbb{Z}_0^-}$, be a sequence such that, $h_k(r + c_k) = f_r, \forall k \in \mathbb{Z}_0^-$, where, $f_r\,\text{fixed} \in S$ and, $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$

Propositions:

P1. The functions, $h_k(y)$, are C^p smooth, $\forall k \in \mathbb{Z}$

P2. $\forall \varepsilon > 0, \exists N \in \mathbb{Z}_0^-$, such that, $\|h_k(y + c_k) - S\| < \varepsilon, \forall k \le N$, where, $\|h_k(y + c_k) - S\| = \sup_{n \in \mathbb{N}[0,q]} |h_k(n + c_k) - f_n|$

P3. $h(n + c) = \mathcal{K}_m[u](n + c) = f_n, \forall f_n \in S$

P4. The limit constant, $c = c(f_0)$, is independent of the choice of r

Proof

P1. Let, $s_k(y) = \mathop{\mathrm{C}}_{j \in \mathbb{Z}[k, +\infty)}^{t} [m(y - j)u(t)]$.

Without loss of generality we can assume that, $k \le 0$, then:

$$h_k(y) = u^{[k]} \circ s_k(y) = u^{[k]} \circ \mathop{\mathrm{C}}_{j \in \mathbb{Z}[k, -1]}^{t} [m(y - j)u(t)] \circ s_0(y) = u^{[k]} \circ \mathop{\mathrm{C}}_{j \in \mathbb{Z}[k, -1]}^{t} [m(y - j)u(t)] \circ h_0(y)$$

Thus, $h_k(y)$, is C^p smooth since it is a composition of C^p smooth functions.

P2. Either, $r-1\in \mathbb{Z}[0,q]$, or, $r+1\in \mathbb{Z}[0,q]$. Without loss of generality we can assume that, $r+1\in \mathbb{Z}[0,q]\Rightarrow f_{r+1}\in S$. We will show that, $h_k(r+1+c_k)-f_{r+1}\to 0$, as, $k\to -\infty$.

- Let:
  $\delta_k = h_k(r+1+c_k)-f_{r+1} = u^{[k]}\circ s_k(r+1+c_k)-f_{r+1} =$
  $u^{[k]}\circ s_{k-1}(r+c_k)-f_{r+1} =$
  (since, $s_k(y)=s_{k+q}(y+q)$, by L1,P1)
  $u^{[k]}\circ (m(r-(k-1))u\circ s_k(r+c_k))-f_{r+1} =$
- $u^{[k]}\circ (m(r-k+1)u\circ u^{[-k]}\circ u^{[k]}\circ s_k(r+c_k))-f_{r+1} =$
  $u^{[k]}\circ (m(r-k+1)u^{[1-k]}(f_r))-f_{r+1}\Rightarrow$
  (since, $u^{[k]}\circ s_k(r+c_k)=f_r$)
  $u^{[k]}\circ (m(r-k+1)u^{[1-k]}(f_r)) = \delta_k + f_{r+1}\Rightarrow$
- $u^{[k-1]}\circ (m(r-k+1)u^{[1-k]}(f_r)) = u^{-}(\delta_k+u(f_r))\Rightarrow$
  $\lim\limits_{k\to-\infty}\left[u^{[k-1]}\circ (m(r-k+1)u^{[1-k]}(f_r))\right] = f_r\Rightarrow$
  (since, $\lim\limits_{k\to-\infty} m(r-k+1)=1$)
- $\lim\limits_{k\to-\infty} u^{-}(\delta_k+u(f_r)) = f_r \Rightarrow \lim\limits_{k\to-\infty}\delta_k = 0\Rightarrow$
  (since *u*,is strictly increasing)
  $\lim\limits_{k\to-\infty} h_k(r+1+c_k) = f_{r+1}\Rightarrow \lim\limits_{k\to-\infty} c_k = c\Rightarrow$
  $h(r+1+c) = \mathcal{K}_m[u](r+1+c) = f_{r+1}$

In a similar way we can show that, $\lim\limits_{k\to-\infty} h_k(n+c_k) = f_n, \forall f_n\in S$. Since *S*, has a finite number of points the supremum always exists and tends to zero: $\sup\limits_{n\in\mathbb{Z}[0,q]} |h_k(n+c_k)-f_n|\to 0$. Then, $\forall\varepsilon>0, \exists N\in\mathbb{Z}_0^-$, such that, $\sup\limits_{n\in\mathbb{Z}[0,q]} |h_k(n+c_k)-f_n| = \|h_k(y+c_k)-S\| < \varepsilon$.

P3, P4. P2 immediately implies that, $h(n+c)=\mathcal{K}_m[u](n+c)=f_n, \forall f_n\in S$. This again immediately implies that *c*, is independent of the choice of *r* and for any particular *u*, depends only on, f_0 and *m*.

4.2. The Functional Transform, $\mathcal{K}_m[u]$, as a Solution of (1)

Theorem 5: Assumptions:

A1. $u\in U$ and, $m\in M$

A2. $h_k(y), \mathcal{K}_m[u](y)\in \mathrm{H}$, where, $y\in\mathbb{R}(y_1,+\infty)$, $y_1\in\mathbb{R}\cup\{-\infty\}$

A3. Let, $c=\lim\limits_{k\to-\infty} c_k$, be the limit translation constant as defined in T4.

Propositions:

P1. The transformed functions, $h(y)=\mathcal{K}_m[u](y)$, satisfy the Abel functional equation, $h(y+1)=u\circ h(y)$, $\forall y\in\mathbb{R}(y_1,+\infty)$.

P2. The functional transform, $\mathcal{K}_m[u](y+c)$, is a flow of (1) through f_0 and thus completely interpolates the orbit $O(f_0)$.

Proof

P1. We have:

$$h_k(y+1) = u^{[k]}\circ \mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,+\infty)}^{t}[m(y+1-j)u(t)] =$$

$$u^{[k]}\circ \mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k-1,+\infty)}^{t}[m(y-j)u(t)] = \qquad \text{(by L1,P1)}$$

$$u\circ u^{[k-1]}\circ \mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k-1,+\infty)}^{t}[m(y-j)u(t)] = u\circ h_{k-1}(y)\Rightarrow$$

$$\lim_{k\to-\infty} h_k(y+1) = \lim_{k\to-\infty} u\circ h_{k-1}(y)\Rightarrow h(y+1)=u\circ h(y)$$

And this proves P1.

P2. Since, $\mathcal{K}_m[u](c)=f_0$, the proposition P2 follows from P1. Q.E.D.

4.3. General Solution of the Abel Functional Equation, *f(x + 1) = u ∘ f(x)*

For, $u\in U$, the functional transform delivers particular solutions of the Abel FE, $f\circ S = u\circ f$, where *S*, is the successor function. The following known lemma gives a general solution of the AFE, from any particular solution, where it is not necessary that the particular solution is derived from the functional transform.

Lemma 6: Assumptions:

A1. *I*, is the identity function, *S*, is the successor function and, $\varphi:\mathbb{R}\to\mathbb{R}, \Phi:\mathbb{R}\to\mathbb{R}, u:D\to D\subset\mathbb{R}$, $f:A\subset\mathbb{R}\to\mathbb{R}$, are continuous functions.

A2. Consider the FEs:
$f\circ S = u\circ f$ (1) Abel FE
$\Phi\circ S = S\circ\Phi$ (2) FE of diagonally 1-periodic functions

$\varphi \circ S = I \circ \varphi$ (3) FE of 1-periodic functions

where the domain *A*, of *f*, is such that FE (1) is completely satisfied on the respective domains.

Propositions

P1. The general solution of (1) is, $h = f \circ \Phi$, where, $f: A \to \mathbb{R}$, is a particular continuous solution of (1) and Φ, is an arbitrary solution of (2).

P2. The general solution of (2) is, $\Phi = I + \varphi$, where φ, is an arbitrary solution of (3). We call Φ, a diagonally 1- periodic function.

P3. The general solution of (3) is an arbitrary 1-periodic function φ.

Proof

P1. We have:

$$h \circ S = f \circ \Phi \circ S = f \circ S \circ \Phi = u \circ f \circ \Phi = u \circ h$$

P2. We have:

$$\Phi \circ S = (I + \varphi) \circ S = I \circ S + \varphi \circ S = S \circ I + \varphi = S \circ (I + \varphi) = S \circ \Phi$$

where, we have used the following property of the successor function: $S \circ (u + v) = S \circ u + v = u + S \circ v$.

P3. $\varphi \circ S = I \circ \varphi = \varphi$, is the definition of 1-periodic functions

In assumption A2 we have included the identity function in FE (3), to stress the elegant symmetry of the three functional equations. This theorem can easily be extended to wider classes of functions by taking care of the respective domains and image sets.

Notice that the related functional equation, $g \circ u = S \circ g$, is also called the Abel FE. Actually this was the original FE considered by Abel (1881).

4.4. Existence of Smooth Solutions of (1)

The following lemma is an adaptation of the main theorem stated by (Belitskii and Lyubich, 1999).

Lemma 7: Assumptions:

A1. *f*, satisfies the AFE: $f \circ S = u \circ f$ (1).

A2. $u \in U$

A3. *u*, is C^k smooth where, $k \in \mathbb{N} \cup \{+\infty, \omega\}$, where C^ω smooth means real analytic.

A4. If, $k \geq 1$, let, $(f^-)' \neq 0$.

Propositions:

P1a. If, $k = 0$, there exists a continuous solution of (1)

P1b. If, $k \geq 1$, there exists a C^k diffeomorphic solution of (1)

Proof

P1. We consider the FE: $g \circ u = S \circ g$. Assumption A2 implies that *u*, has no real fixed points and this implies that every compact subset of, $A \subset \mathbb{R}$, is wandering under *u*, which means that, $\exists N \in \mathbb{N}$, such that for $p, q \in \mathbb{N}$, we have: $u^{[p]}(A) \cap u^{[q]}(A) = \phi, \forall p - q \geq N$, see Belitskii and Lyubich (1999). From the main result in (Belitskii and Lyubich, 1999) we have that in this case there exists an invertible solution of, $g \circ u = S \circ g$, which is C^k smooth. Let *g*, be such a solution. Since *g*, is invertible let, $f = g^-$. Then: $f^- \circ u = S \circ f^- \Rightarrow f \circ S = u \circ f$. Thus *f*, is a solution of (1)

For, $k = 0$, *g*, is continuous thus, $f = g^-$, is continuous. For, $k \geq 1$, *g*, is $C^{k\,k}$ smooth and the chain rule implies that *f*, is C^k smooth provided that, $(f^-)' \neq 0$ (A4). Thus *f*, is a C^k diffeomorphism.

Thus we have established that C^k smooth solutions of (1) exist. We have not proved in this article that the functional transform, $\mathcal{K}_m[u]$, indeed delivers these C^k smooth solutions for appropriate modulator functions. Nevertheless we have strong evidence that this is the case.

4.5. Homologous and Principal Functions

The general solution in Lemma 6 of the AFE (1) with respect to *u*, defines a class H_u, of continuous functions:

$$H_u = \{f \mid f, \text{is a solution of the AFE (1) in L6}\}$$

We call any two functions in H_u, homologous functions. By L6 any two homologous functions f_1, f_2 are related as: $f_1 = f_2 \circ \Phi$, where Φ is a real diagonally 1- periodic function. The question arises whether there is any unique privileged function in H_u, called the principal function of H_u. Kuczma *et al*. (1990) describes such principal solutions of the AFE in the sense of (Szekeres, 1958), which however do not apply to functions of the class *U*, as defined in section 3. Nevertheless we have found strong evidence (not presented in this article) that unique privileged functions exist for functions of the class *U*. These functions define principal solutions of the dynamical system defined by *u*.

In some cases of AFE with known principal solutions, we have found an appropriate modulator function *m*, (called a principal modulator function that corresponds to *u*) such that, $\mathcal{K}_m[u]$, is identical with the principal solution. The principal modulator function corresponding to each *u*, is probably not unique. A most surprising property though of many functional transforms, $\mathcal{K}_m[u]$, was that by using a variety of simple modulator functions, we got homologous solutions that are extremely close to the principal solution. Indeed if, $h = f \circ \Phi$, is a homologous solution and *f*, is the principal solution we have found that the amplitudes of Φ, are of the order of 10^{-5} to 10^{-13}. By amplitude of Φ, we mean the amplitude of the 1 - periodic function, $\varphi(x) = \Phi(x) - x$.

5. APPLICATIONS: MATHEMATICAL AND PHYSICAL

5.1. Continuous Iterates of Functions

The functional transform method presented in this article has both mathematical and physical applications.

In the area of mathematics it is an approach to define continuous iterates of functions. It is known that for continuous strictly increasing functions *u*, there is always a subclass of homologous continuous strictly increasing functions *h*, which are solutions of the AFE (1). With the help of these solutions we can give a rigorous definition of continuous iterates of a function *u*. Following the discussion in subsection 4.5 about homologous functions, we propose a definition of continuous iterates as:

principal iterates: $u^{[y]}(x) \equiv f(y + f^{-}(x))$

homologous iterates: $u_{\Phi}^{[y]}(x) \equiv h(y + h^{-}(x))$

where, *f*, is the principal solution of (1), *h*, is a homologous solution and Φ, is a diagonally 1-periodic function such that, $h = f \circ \Phi$. A principal iterate is also a homologous iterate, but not vice versa. If we define, $\Phi(n) = 0, \forall n \in \mathbb{Z}$, all homologous iterates are identical with the principal iterate at integer values of *y*. For non-integer *y*, the homologous iterates generally have different values (for differentiable Φ, they may have identical values only on a set of measure zero). In our opinion the functional transform is a significant step towards the use of homologous iterates, since it substantially expands the class of functions *u*, for which a solution of (1) is available.

5.2. Physical Applications

We describe a general physical experiment where the functional transform could be usefull. In this experiment we measure the input f_0 and the output f_1, of a physical quantity at time $t = 0$ and $t = 1$, respectively. We make following assumptions:

- The experiment can be performed repeatedly
- The output f_1, depends deterministically on the input f_0, which implies that for identical inputs we get identical outputs
- The input and output vary continuously and from the measurements we can estimate a continuous function *u*, that governs the dynamical system, $f_1 = u(f_0)$
- We assume that the dynamical system is controlled by *u*, for a time period *T*
- The function *u*, belongs to the class *U*, described in 3.1

Then the functional transform, $\mathcal{K}_m[u]$, is a continuous solution of the dynamical system and gives the quantity, $f(t)$, at any time, $t \in T$.

Next we give an example to demonstrate the method.

An Example: $u(x) = x^2 + 1$

We demonstrate our method on the Dynamical System (DS), $f_{n+1} = (f_n)^2 + 1$, hence the defining map is, $u : D \to \mathbb{R}$, $D = \mathbb{R}\,(0, +\infty)$, $u(x) = x^2 + 1$, $u^{-}(x) = \sqrt{x-1}$. For this example we chose the logistic function, $m(x) = \dfrac{1}{1+e^{-x}}$, as the modulator function and we seek a smooth solution for the starting value, $f_0 = 0$. With the functional transform method, we will approximate as close as possible (considering the limitations of our computer) the values f_r, $r = 1,2,3,4,5,6$, of the orbit of this DS, assuming only the starting value, $f_0 = 0$. The first few exact values of the orbit are given by the integers:

f_0	0
f_1	1
f_2	2
f_3	5
f_4	26
f_5	677
f_5	458 330

Applying the functional transform method to *u*, we have:

$$h_{k,n}(y,x)=u^{[k]}\circ \mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t}[m(y-j)u(t)]\circ(x)\Rightarrow$$

$$h_{k,n}(y,1)=\left(\sqrt{t-1}\right)^{[k]}\circ \mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,n]}^{t}\left[\frac{t^2+1}{1+e^{-(y-j)}}\right]\circ(1)\Rightarrow$$

$$h_k(y)=\left(\sqrt{t-1}\right)^{[k]}\circ \mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,15]}^{t}\left[\frac{t^2+1}{1+e^{-(y-j)}}\right]\circ \mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[16,+\infty]}^{t}\left[\frac{t^2+1}{1+e^{-(y-j)}}\right]$$

We have checked that using the first 15 terms of the continued form is sufficient for our purposes, since the later terms contribute only about 10^{-30} to the result. Of course in principle one can take any number of terms. First we will determine the translation constants c_k, for, k = -1, -2,…, -31, for the chosen orbit point, $f_0=0$, (which implies, y = 0). We use the equation:

$$h_k(0+c_k)=h_k(c_k)\square$$

$$\left(\sqrt{c_k-1}\right)^{[k]}\circ \mathop{\mathrm{C}}\limits_{j\in\mathbb{Z}[k,15]}^{t}\left[\frac{t^2+1}{1+e^{-(c_k-j)}}\right]\circ(1)\square\ f_0=0$$

Below we give the convergents c_k, showing only the correct decimal digits c_{dig}, for various k:

k	c_k	c_{dig}
-6	0.3208	4.4
-11	0.32086284	8.1
-16	0.32086284925	11.8
-21	0.320862849249683	15.5
-26	0.3208628492496829344	19.1
-31	0.3208628492496829344748	22.8

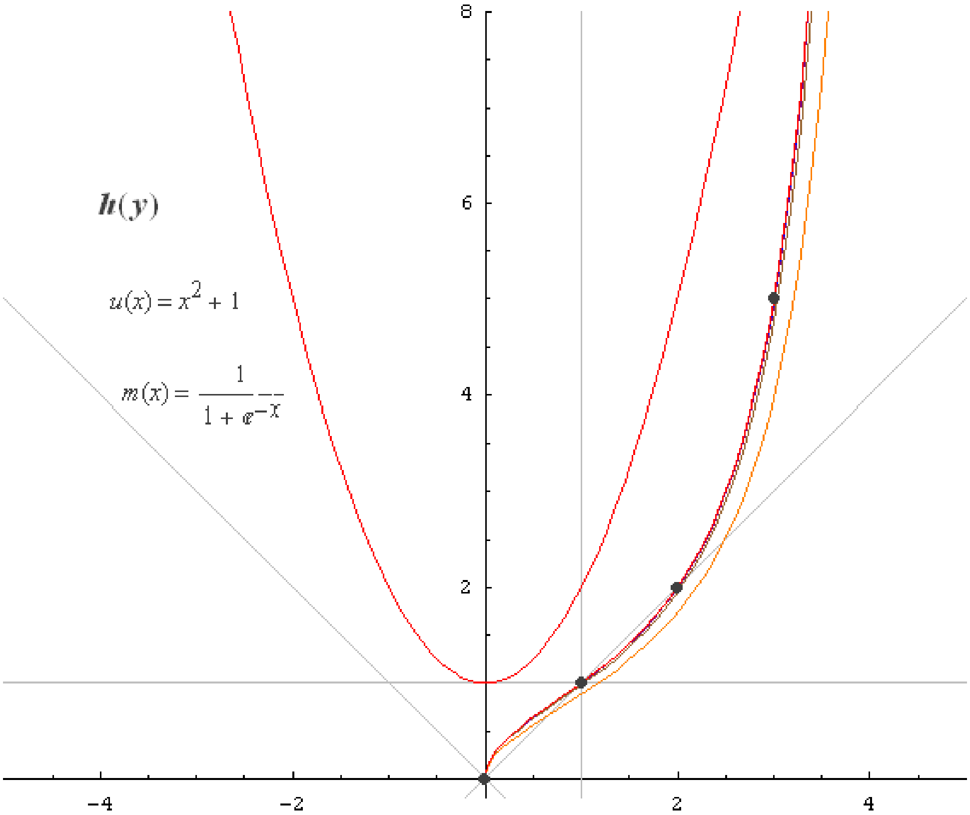

Fig. 1. Interpolation of the orbit points of, $u^{[n]}(0), n=0,1,2,3$, by the convergents, $h_k(x)$, where, $k=-1,-2,-3,-4,-5$, (Section 5.2). At the limit, $h_k(x)\to h(x)=\mathcal{K}_m[u](x)$

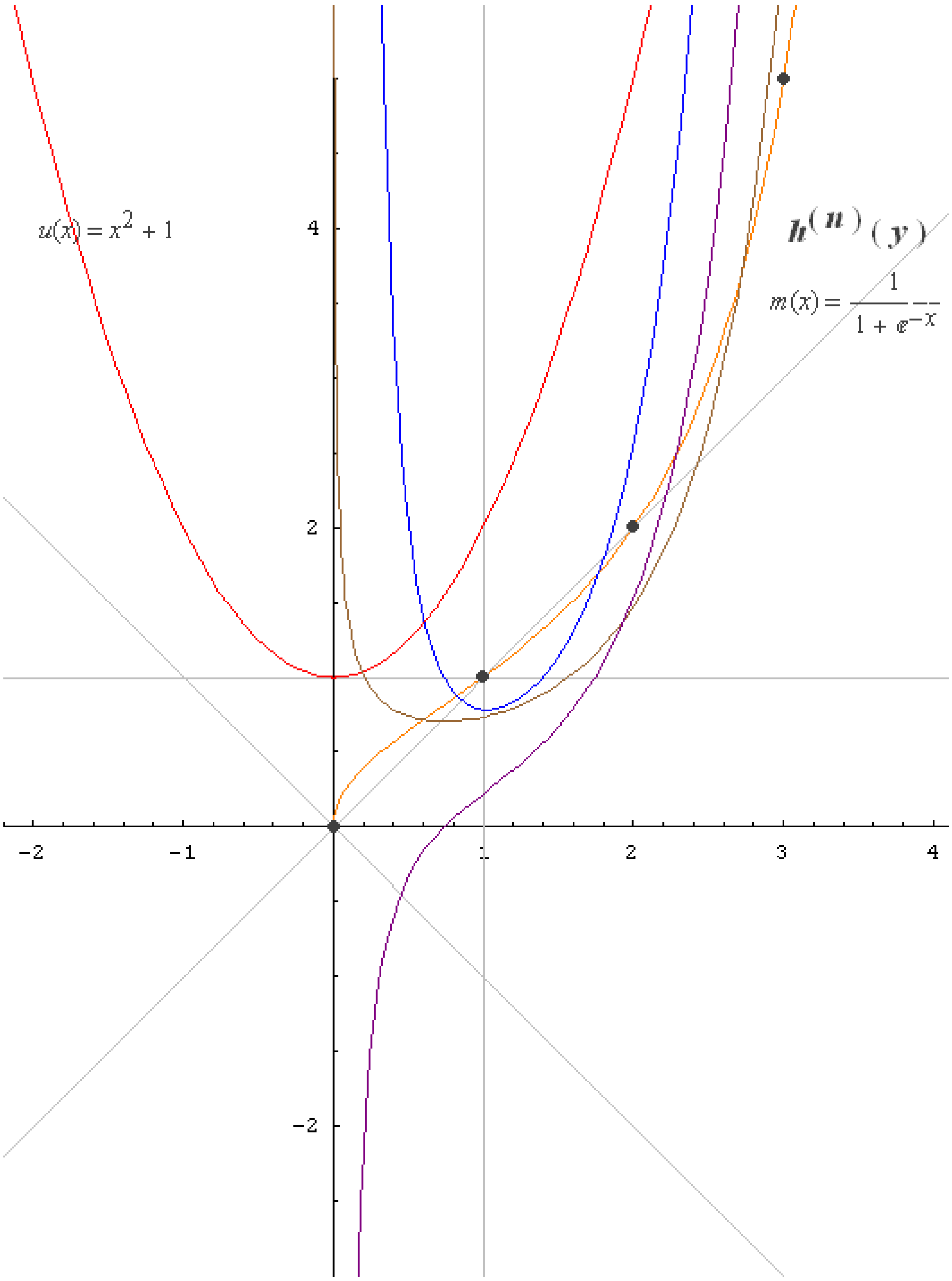

Fig. 2. First three derivatives (brown, purple, blue) of the functional transform, $h(x) = \mathcal{K}_m[u](x)$, (orange). (Section 5.2)

Clearly c_k, approaches a limit value. The correct digits increase linearly with $|k|$, according to, $c_{dig} \approx 0.735|k|$.

Next we give the difference between the function values and the orbit points, $\Delta_{r,k} = f_r - h_k(r + c_k)$, where, $r = 1,2,3,4,5,6$, for various k.

$k \backslash r$	1	2	3	4	5	6
-6	2.7×10^{-5}	6.4×10^{-5}	2.6×10^{-4}	2.6×10^{-3}	0.14	190
-11	5.7×10^{-9}	1.3×10^{-8}	5.5×10^{-8}	5.5×10^{-7}	2.9×10^{-5}	3.9×10^{-2}
−16	1.2×10^{-12}	2.8×10^{-12}	1.2×10^{-11}	1.2×10^{-10}	6.1×10^{-9}	8.2×10^{-6}
−21	2.5×10^{-16}	6.0×10^{-16}	2.4×10^{-15}	2.5×10^{-14}	1.3×10^{-12}	1.7×10^{-9}
−26	5.3×10^{-20}	1.3×10^{-19}	5.2×10^{-19}	5.2×10^{-18}	2.7×10^{-16}	3.7×10^{-13}
−27	9.7×10^{-21}	2.3×10^{-20}	9.5×10^{-20}	9.5×10^{-19}	5.0×10^{-17}	Overflow
−28	1.8×10^{-21}	4.2×10^{-21}	1.7×10^{-20}	1.8×10^{-19}	Overflow	Overflow
−29	3.3×10^{-22}	7.8×10^{-22}	3.2×10^{-21}	Overflow	Overflow	Overflow
−30	6.1×10^{-23}	1.4×10^{-22}	Overflow	Overflow	Overflow	Overflow
−31	1.1×10^{-23}	Overflow	Overflow	Overflow	Overflow	Overflow

We see that, $h_k(r+c_k)$, converges fast to, f_r, as *k*, decreases, until the computation is stopped by overflow. This is a computational difficulty that eventually will be met on all computers. There are some methods (not mentioned in this article) to temporarily overcome this problem, which add a few more steps at the cost of rapidly increasing complexity of calculations. **Figure 1** shows how fast, $h_k(y+c_k)$, approaches the orbit points of *u*.

It is straightforward to compute higher derivatives of the functional transform **Fig. 2** shows the first three derivatives. From these we get the first few terms of the series expansion at, $y=1$:

$$\mathcal{K}_m[x^2+1](y)=h(y)=1+0.735198(y-1)+$$
$$0.221822\frac{(y-1)^2}{2}+0.783798\frac{(y-1)^3}{3!}-0.10186\frac{(y-1)^4}{4!}+\ldots$$

With this example we have demonstrated that the functional transform may seem complicated to calculate, but with the computer it is in principle no more difficult than the calculation of the exponential function from its Taylor series. The main obstacle we encountered in calculating functional transforms of various functions was the overflow barrier and only seldom the time limitation of calculations was *a* problem. For example the highest order translation constant c_{-31}, of our sample functional transform, $\mathcal{K}_m[x^2+1]$, needed only a split second to calculate with 23 accurate decimal digits on an average PC, before it was interrupted by overflow. In the future the overflow barrier will be pushed to much higher numbers. This means that for *a* large subclass of functions in the class *U*, the functional transform will be computable almost instantly to any desired precision. In turn this will make the functional transform and the homologous iterates even more usefull.

We have calculated the functional transform for various smooth functions *u*, with simple and complicated rules. In all cases the functional transform method has delivered quite smooth solutions for which we have in cases calculated and plotted the derivatives in excess of the 10 th order. The evidence is that the functional transform method delivers very smooth solutions.

Form the physical perspective the functional transform presented in this article applies to dynamical systems that increase to infinity. Since infinities are not directly observable in the universe, we must expect that there will be *a* sudden break down of the dynamical system at some point. Nevertheless it is perfectly legitimate to apply the continuous solution for the time period before the break down. The fact that the functional transform uses internally an extrapolation to infinity does not affect the solution for the period when the dynamical system is valid. Contrariwise this extrapolation facilitates or at least greatly .improves the convergence of the functional transform.

6. DISCUSSION

The functional transform delivers smooth solutions for dynamical systems governed by the Abel FE, $f\circ S=u\circ f$, where, $u\in U$, is defined in Section 3.1. A large number of known results deals with particular functions *u* (ex. Hooshmand, 2006). Although a few methods are available which deliver solutions for more general classes of *u*, (most deal with the related original Abel FE, $f^{-}\circ u=S\circ f^{-}$), none of the previous results known to us, directly applies to strictly increasing functions without fixed points with, $u(x)>x$. Via these solutions we can determine the iterative roots of functions in *U*, which are usefull in many applications. See Baron and Jarczyk (2001) for a survey on iterative roots and the AFE.

The functional transform presents a novel method using continued forms for the determination of solutions of functional equations, which to our knowledge has not been previously used. Moreover the continued forms themselves, which represent arbitrary successive compositions, may prove a handy notation for a variety of mathematical settings where a large number of repeated compositions is required.

7. OPEN PROBLEMS

We give a brief description of some open problems related to this article:

- Provided that *u*, is C^p smooth, a rigorous proof is required that the functional transform delivers at least C^p smooth functions, $\forall p \in \mathbb{N} \cup \{+\infty, \omega\}$. A counterexample would also be usefull
- A rigorous definition of principal functions in the set of homologous functions H_u, is needed where *u*, is as in Lemma 6
- Finding ' principal ' modulator functions corresponding to each *u*, that deliver the principal solutions
- Extension of the class *U*, for which the functional transform applies and defining the class for which it delivers smooth solutions. Also extension of the class of modulator functions *M*

8. CONCLUSION

The functional transform method delivers sufficiently smooth flows of the dynamical system, $f_{n+1} = u(f_n)$, for each initial value f_0, for a class of strictly increasing real functions, $u \in U$. The flows depend on an arbitrary modulator function. In this article we have demonstrated the method and proved some important theorems which show that the method works. As a sideproduct we have introduced the notion of continued forms which represents successive composition of functions and is *a* generalization of continued fractions. The proofs of the theorems presented in the current article were significantly more clear due to the use of continued forms.

In following articles we intend to expand the class of functions *U*, for which the functional transform method can be applied. We also intend to give more applications both mathematical and physical.

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