Journal of Mathematics and Statistics

Original Research Paper

A Simple and Accurate Relation Between the Logarithm Integral *Li*(*x*) and the Primes Counting Function π(*x*) is Derived Making use of the O.E.I.S. Prime Numbers "Sequences"

Paolo Ascarelli

Department of Material Science, C.N.R retired, Italy

Article history
Received: 22-07-2021
Revised: 31-08-2021
Accepted: 04-09-2021

Email: ascarelli.p@gmail.com

Abstract: Today the prime numbers $\pi(x)$ contained under the number x appears to be somewhat overestimated by the logarithm integral function $Li(x)$ and underestimated by the function $x/\ln(x)$, both originally proposed by Gauss around 1792-1796. However, a simple and accurate expression, relating $Li(x)$ and $\pi(x)$, may be derived using the data reported on the O.E.I.S. "Sequences". This relation can also suggest the possibility that for very big numbers the $Li(x)$ may oscillate around $\pi(x)$.

Keywords: Numbers Theory, Mathematics History in Grammar School

Introduction

Gauss, around the period 1792-1796, examining and ordering the data on prime numbers available to him, conjectured that the primes counting, defined as the number of primes occurring under the number (x), could be approximated by the expression:

$$\pi(x) = x/\ln(x). \quad (1)$$

But shortly after he suggested a more precise expression which he supported by a deeper mathematical observation. He approached the problem as it was a probability calculation because he observed that the primes density was decreasing the bigger was their number (x). Therefore, he considered the following expression $\pi(x)$= 1/ln(2) + 1/ln(2) + 1/ln(3) +…+ 1/ln(x)= $\sum_{i=2}^{x} 1/\ln(i)$ and he proposed what he called the logarithm integral:

$$Li(x) = \int_2^x (dy/\ln(y)). \quad (2)$$

He was convinced that, as the second expression was more accurate than the first one, the $Li(x)$ would always lightly overestimate $\pi(x)$, while $x/\ln(x)$ underestimate $\pi(x)$.

That expression has not shown any exception until today $Li(x) \geq \pi(x)$.

However, J.E. Littlewood (1914) proved and, later many others, demonstrated that eventually the $Li(x)$ for very big (x) would oscillate an infinite number of times crossing the function $\pi(x)$. Nevertheless, the smaller value of (x), where this phenomenon was predicted to occur, was too big to be verified by the modern technology.

Discussion

In this note is shown that a simple and accurate expression relating $Li(x)$ to $\pi(x)$ can be derived using the data of the On-Line-Encyclopedia of Integer "Sequences" O.E.I.S.

This relation, which is reported in column 4 in Table 1, also suggests that $Li(x)$ may eventually oscillate around $\pi(x)$.

It can be noticed that the development of the function ($Li(x)$ - $\pi(x)$) is easily and accurately represented by the square–root of the function $(\pi(x) - x/\ln(x))^{1/2}$.

That is:

$$Li(x) - \pi(x) = (\pi(x) - x/\ln(x))^{1/2} \quad (3)$$

This behavior is starting at $x = 10^9$, where the trend of the function appears to be stabilized and is reported on the table up to the number $x = 10^{23}$. This seemed sufficient to evidence the relation between the two functions. The Eq. 3 can also be proposed as:

$$(Li(x) - \pi(x))^2 = (\pi(x) - x/\ln(x)). \quad (4)$$

This suggests to consider the be-quadratic function:

$$\left(Li(x)\right)^2 - 2\pi(x)Li(x) + \left(\pi(x)\right)^2 - \left(\pi(x) - \frac{x}{\ln(x)}\right), \qquad (5)$$

which can be solved providing two possibilities:

$$Li(x) - \pi(x) = \left(\pi(x) - x/\ln(x)\right)^{1/2} \qquad (6)$$

$$\pi(x) - Li(x) = \left(\pi(x) - x/\ln(x)\right)^{1/2} \qquad (7)$$

The results of the Eq. 6 have been already reported in column 4 in Table 1 . Instead, Eq. 7 may suggest the possible occurrence of "oscillations" of the function Li(x) around $\pi(x)$. In reality it may be observed, as it is shown in the last column on the table, that the results of Eq. 6 represent the "errors" of the function $\pi(x)$ to reach the $x/\ln(x)$, which is known to be the limit of $\pi(x)$. This is demonstrated in the table by the results of the following Eq. 8:

$$\left(\pi(x) - x/\ln(x)\right)^{1/2} = \left(\frac{\pi(x)}{\left(\ln(x)\right)^2}\right)^{1/2}\left(1 + \frac{2}{\ln(x)}\right)^{1/2}. \qquad (8)$$

Conclusion

Thus, this note has demonstrated the possible occurrence of the "oscillations" of $Li(x)$ around $\pi(x)$, but has not proven that they exist. On the contrary Littlewood (1914) has suggested, making use of the Riemann function R(x), that the "oscillations" exist. But this may raise some questions, considering also the results of the work of Kotnik (2008), who concluded that the Riemann function does not allow a better formulation of the function $Li(x)$.

Acknowledgement

The author is member of the American Physical Society since 1966

References

Kotnik, T. (2008). The prime-counting function and its analytic approximations. Advances in Computational Mathematics, 29(1), 55-70. https://link.springer.com/article/10.1007/s10444-007-9039-2

Littlewood, J.E. (1914). Sur la distribution des nombres premiers. Comptes rendus Acad. Sci. Paris, 158 (1914), 1869-18972

Table 1: O.E.I.S. "Sequences" with results by Eq. 6 and Eq. 8

x= 10^n	(Li(x)-π(x) A057752 O.E.I.S.	π(x)- x/ln(x) A057835 O.E.I.S.	(Li(x)-π(x))/(π(x) -(x/ln(x)))^1/2 *	(π(x) -x/ln(x))^1/2	((x^1/2)/ln(x)) *(1+2/ln(x))^1/2
2	5	3	2,886	1,81	
3	10	23	2,085	4,79	
4	17	143	1,421	11,95	
5	38	906	1,161	30,09	
6	130	6116	1,662	78,2	
7	339	44158	1,613	210,1	
8	754	332774	1,307	576,8	
9	1701	2592592	1,0564	1610	1672
10	3104	20758029	0,68128	4556	
11	11589	169923159	0,88899	13935	
12	38263	1416705193	1,01657	37639	37478
13	108971	11992858452	0,99506	109511	
14	314890	102838308636	0,98193	320684	
15	1052519	891604962452	1,11476	944248	
16	3214632	7804728884393	1,15070	2793694	2787042
17	7956589	68883734693281	0,96682	822922	
18	21949555	612483070893536	0,88691	24748395	
19	99877775	5481624169369960	1,34900	74637991	
20	222744644	49347193044659701	1,00271	222142281	221812420
21	597394254	446579871578168707	0,89394	668266317	
22	1932355208	4060704006019620994	0,95892	2015118857	
23	7250186216	37083513766578631308	1,19058	6089623450	

*Notice: The average of the numbers in this column is 1.00281, starting from the number 9 where stabilization of the trend is observed