Original Research Paper

q-Deformation of Transonic Gas Equation

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Abstract: In this study, the transonic gas equation will be considered. For $q \in (0,1)$, q-deformation of transonic gas equation (*q-transonic*) are studied, we use q- derivative (or Jackson derivative) to solve transonic gas equation.

Keyword: Transonic Gas Equation, q-Deformation, q-Derivative

Introduction

In real complex and functional analysis, derivatives are generalized to functions of several real or complex variables and functions between topological vector spaces. An important case is the variational derivative in the calculus of variations. Repeated application of differentiation leads to derivatives of higher order and differential operators. The derivative is often met for the first time as an operation on a single real function of a single real variable. One of the simplest settings for generalizations is to vector valued functions of several variables (most often the domain forms a vector space as well). This is the field of multivariable calculus. In one-variable calculus, we say that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a point x if the limit $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists. Roughly speaking, in mathematics, specifically in the areas of combinatorics and special functions, a q-analog of a theorem, identity or expression is a generalization involving a new parameter q that returns the original theorem, identity or expression in the limit as $q \to 1$ (this limit is often formal, as q is often discrete-valued). Typically, mathematicians are interested in q-analog that arise naturally, rather than in arbitrarily contriving q-analog of known results. The earliest q-analog studied in detail is the basic hypergeometric series, which was introduced in the 19th century. q-analog find applications in a number of areas, including the study of fractals and multi-fractal measures and expressions for the entropy of chaotic dynamical systems. The relationship to fractals and dynamical systems results from the fact that many fractal patterns have the symmetries of Fuchsian groupssee (Accardi and Boukas, 2006; Accardi *et al*., 2006; Ettaieb *et al*., 2014; Leeuwen and Maassen, 1995; Hida, 1993). The connection passes through hyperbolic geometry and ergodic theory, where the elliptic integrals and modular formsplay a prominent role; the q-series themselves are closely related to elliptic integrals. q-analogs also appear in the study of quantum groups and in q-deformed superalgebras. The connection here is similar, in that much of string theory is set in the language of Riemann surfaces, resulting in connections to elliptic curves, which in turn relate to q-series. There are two main groups of q-analogs, the "classical" q-analogs, with beginnings in the work of Leonhard Euler and extended by F.H. Jackson and others. q-analogs are often found in physical applications, especially exact solutions of many-body problems. In such cases, the $q \to 1$ limit usually corresponds to relatively simple dynamics, e.g., without nonlinear interactions, while $q<1$ gives insight into the complex nonlinear regime with feedbacks. An example from atomic physics is the model of molecular condensate creation from an ultra-cold fermionic atomic gas during a sweep of an external magnetic field through the Fischbach resonance. This process is described by a model with a q-deformed version of the SU(2) algebra of operators (The special unitary group SU(n) is a real Lie group though not a complex Lie group) (Accardi and Boukas, 2009; Bangerezako, 2004; Bożejko and Speicher, 1991; Ettaieb *et al*., 2012; Jackson, 1910; Mason, 1915) and its solution is described by q-deformed exponential and binomial distributions. This article is organized as follows: In section 2 we state the main problem transonic gas equation, in section 3 introduced q-derivative, in section 4 introduced some q-calculus definitions related to q-deformation and Jackson derivative, in section 5 the classical solution of transonic gas equation presented. Finally used to q-deformation to deduce a solution of transonic gas equation.

Transonic Gas Equation

Consider the transonic gas equation:

$$u_x u_{xx} + u_{yy} = 0. \tag{1}$$

The study of gas dynamics is often associated with the flight of modern high-speed aircraft and atmospheric reentry of space-exploration vehicles; however, its origins lie with simpler machines. At the beginning of the 19th century, investigation into the behavior of fired bullets led to improvement in the accuracy and capabilities of guns and artillery. As the century progressed, inventors such as Gustaf de Laval advanced the field, while researchers such as Ernst Mach sought to understand the physical phenomenon involved through experimentation the model equation gas equation. In classical solution of transonic gas flow the solution take a system of equations but the number of equations is less the number of unknowns parameters, for this reason we cannot have a unique solution for our differential equation. This leads to an open question, what can we do in our case or how can we change completely the PDEs in order to get a solvable system with unique solution. So in this study we solve the transonic gas equation using q-deformation.

q-Derivative

Here is a nice diversion for anyone who knows what the derivative of a simple function is $f(x)$. The modern theory of differential and integral calculus began in the 20th century with the works of Newton and Leibniz. As it is well known, the derivative of a function $f(x)$ with respect to the variable x is by definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Now, let us consider the following expression:

$$f'(x) = \lim_{q \to 1} \frac{f(qx) - f(x)}{qx - x}.$$

Of course, this is not valid when $q = 1$ or $x = 0$ but otherwise this alternative formula is equivalent to the usual derivative. You can convince yourself by writing $\dfrac{f(x + (q-1)x) - f(x)}{(q-1)x}$, the term $(q-1)x$ playing the role of h.

At the beginning of the 20th century, F.H. Jackson studied this modified derivative and many of its consequences. The key concept is the q-derivative operator defined as follows when $0<q<1$:

$$\left(D_q f\right)(x) = \lim_{q \to 1} \frac{f(qx) - f(x)}{qx - x},$$

this q-derivative can be applied to functions not containing 0 in their domain of definition. Then it reduces to the ordinary derivative when q goes to 1:

$$\lim_{q \to 1} \left(D_q f\right)(x) = f'(x).$$

Example

Compute the q-derivative of $x^2 + 2x + 1$:

$$D_q\left(x^2 + 2x + 1\right) = \frac{\left[(qx)^2 + 2(qx) + 1\right] - \left[x^2 + 2x + 1\right]}{qx - x}$$
$$= (1-q)x + 2.$$

One can easily check that the q-derivative operator is linear:

$$D_q\left(f + g\right) = D_q f + D_q g$$
$$D_q\left(\lambda f\right) = \lambda D_q f,$$

the product rule is slightly modified but it approaches the usual product rule when q goes to one:

$$\left(D_q(fg)\right)(x) = f(qx)\left(D_q g\right)(x) + \left(D_q f\right)(x)g(x).$$

q-Calculus

We recall some basic notations used in q-calculus (Abdi, 1962; Accardi and Boukas, 2007; Adams, 1929; Altoum *et al.*, 2047; Ettaieb *et al.*, 2016; Hida, 1993; Yuri *et al.*, 1998). The natural number *n* has the following *q* deformation:

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}, \ with [0]_q = 0.$$

Occasionally, we shall write $[\infty]_q$ for the limit of these numbers:

$$\frac{1}{1-q},$$

The *q* factorials and *q* binomial coefficients are defined naturally as:

$$[n]_q! = [1]_q \cdot [2]_q \cdot [3]_q \cdots [n]_q, \ with [0]_q := 0.$$

Recall that from (Yuri *et al.*, 1998), for $q \in (0,1)$ relation $aa^* - qa^*a = 1$ (q-Deformation of the Square

White Noise Lie Algebra) admits, up to unitary equivalence, a unique non-trivial bounded irreducible representation given on the canonical basis:

$$\{e_n | n \in \mathbb{N}\} \, of \, l^2(\mathbb{N})$$

Defined using the following equations:

- $a^* e_n = e_{n+1}$
- $a e_n = [n]_q e_{n-1}$
- $\langle e_n, e_m \rangle = \delta_{n,m}[n]_q!$

For $q \in (0,1)$ and analytic for $f : \mathbb{C} \to \mathbb{C}$ define operators Z and D_q as follows (Ettaieb *et al.*, 2016; Hida, 1993; Yuri *et al.*, 1998):

$$(Zf)(z) := zf(z),$$
$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{z(1-q)}, & z \neq 0 \\ f'(0) \end{cases}.$$

The operator D_q has the following properties:

- $\lim_{q \to 1}(D_q f)(x) = f'(x)$
- $D_q(z^n) = \dfrac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1}$
- $D_q(f(z)g(z)) = (D_q f)(z)g(z) + f(qz)(D_q g)(z)$
- $D_q \dfrac{f(z)}{g(z)} = \dfrac{(D_q f)(z)g(z) - f(z)(D_q g)(z)}{g(z)g(qz)}$

We can apply the Taylor formula to the definition of $(D_q f)(z)$ to get:

$$(D_q f)(z) = \sum_{k=0}^{\infty} \frac{(q-1)^k}{(k+1)!} x^k f^{(k+1)}(x).$$

A q-analog of the Taylor expansion of a function around zero as follows:

$$f(z) = \sum_{n=0}^{\infty} f^n(0)\frac{(z)^n}{(n)!} = \sum_{n=0}^{\infty}(D_q^n f)(0)\frac{z^n}{[n]!}.$$

Classical Solution of Transonic Gas Equation (Titov, 1988)

The exact solution of the transonic gas dynamics given by the Equation 1 is as follows.

Suppose $u(x,t)$ is the solution of the Equation 1. Then the function:

$$u_1 = c_1^{-3}c_2^2 u(c_1 x + c_3, c_2 y + c_4) + c_5 y + c_6,$$

is also a solution, where $c_1, c_2, \ldots, c_6$ are arbitrary constants.

Solution can be investigated are the following:

$$u(x,t) = c_1 xy + c_2 x + c_3 y + c_4$$

$$u(x,t) = \frac{a^2 c_1^3}{39}(y+A)^{13} + \frac{2}{3}a(y+A)^8(x+B)$$

$$+3c_1(y+A)^3(x+B)^2 - \frac{(x+B)^3}{3a(y+A)^2}$$

$$u(x,t) = \frac{(x+c_1)^2}{3a(x+c_1)^3} + c_3 y + c_4$$

$$u(x,t) = -aA^3 y^2 - \frac{B^2}{aA^2}x + c_1 + c_2 \pm \frac{4}{3}(Ax + By + c_3)^{\frac{3}{2}}$$

$$u(x,t) = \frac{1}{3}(Ay+B)(2c_1 x + c_2)^{\frac{3}{2}} - \frac{ac_1^3}{12A^2}(Ay+B)^4 + c_3 y + c_4$$

$$u(x,t) = -\frac{9aA^2}{y+c_1} + 4A\left(\frac{x+c_2}{y+c_1}\right)^{\frac{3}{2}} - \frac{(x+c_2)^3}{3a(y+c_1)^2} + c_3 y + c_4$$

$$u(x,t) = -\frac{3}{7}aA^2(y+c_1)^7 + 4A(x+c_2)^{\frac{3}{2}}(y+c_1)^{\frac{5}{2}}$$

$$-\frac{(x+c_2)^3}{3a(y+c_1)^2} + c_3 y + c_4$$

where, A, B, $C_1, \ldots, C_4$, are arbitrary constant (the first solution is degenerate).

There are solutions of the following forms:

- self-similar solution:

$$z = xy^k, u(x,t) = y^{-3k-2}u(z), z = xy^k$$

where, k is any arbitrary constants.

- Generalized separable solution as:

$$u(x,y) = \phi_1(y) + \phi_2(y)x^{\frac{3}{2}} + \phi_3(y)x^3;$$
$$u(x,t) = \psi_1(y) + \psi_2(y)x + \psi_3(y)x^2 + \psi_4(y)x^3$$
$$u(x,t) = \psi_1(y)\phi(x) + \psi_2(y)$$

Analogous of Transonic Gas Equation

As an *q-analogous* of (1), we introduce *q-transonic* gas equation as follows:

$$(D_{q,x}u)(D_{q,x}^2) + (D_{q,y}^2) = 0, \tag{2}$$

where:

$$D_q f(r) = \begin{cases} \dfrac{f(r) - f(qr)}{r - qr} r \neq 0, & 0 < q < 1, \\ f'(0) \end{cases}$$

then:

$$D_{q,x}f(z_1,z_2) = \begin{cases} \dfrac{f(z_1,z_2)-f(qz_1,z_2)}{z_1-qz_1}, & if z_1 \neq 0, \\ \partial_x f(0,z_2) \end{cases}$$

and:

$$D_{q,y}f(z_1,z_2) = \begin{cases} \dfrac{f(z_1,z_2)-f(z_1,qz_2)}{z_2-qz_2}, & if z_2 \neq 0, \\ \partial_y f(z_1,0) \end{cases}$$

and:

$$D_{q,xx}f(z_1,z_2)$$
$$= \begin{cases} \dfrac{f(z_1,z_2)-2f(qz_1,z_2)-f(q^2z_1,z_2)}{z_1^2-(1-q)^2}, & if z_1 \neq 0, \\ \partial_{xx}f(0,z_2) \end{cases}$$

and:

$$D_{q,yy}f(z_1,z_2)$$
$$= \begin{cases} \dfrac{f(z_1,z_2)-2f(z_1,qz_2)+f(z_1,q^2z_2)}{z_2^2-(1-q)^2}, & if z_2 \neq 0. \\ \partial_{yy}f(z,0) \end{cases} \qquad (3)$$

Now let:

$$D_{q,xy}f(z_1,z_2) = D_{q,x}\big(D_{q,y}F(z_1,z_2)\big)$$
$$= \begin{cases} \dfrac{F(z_1,z_2)-F(qz_1,z_2)}{z_1-(1-q)}, & if z_1 \neq 0 \\ \partial_x f(0,z_2) \end{cases}$$
$$= \begin{cases} \dfrac{f(z_1,z_2)-f(z_1,qz_2)-f(qz_1,z_2)+f(qz_1,qz_2)}{z_1z_2(1-q)^2}, & z_1 \neq 0, z_2 \neq 0 \\ \dfrac{\partial_y f(z,0)-\partial_y f(qz_1,0)}{z_1(1-q)}, & z_1 \neq 0, z_2 = 0 \\ \dfrac{\partial_x f(0,z_2)-\partial_x f(0,qz_2)}{z_1(1-q)}, & z_1 = 0, z_2 \neq 0 \\ \partial_{xy}f(0,0), \end{cases}$$

where, $(z_1 \neq 0 \wedge z_2 \neq 0) \Rightarrow z_1 = 0 \vee z_2 = 0$.
Let:

$$D_{q,x}f(z_1,z_2) = g(z_1,z_2),$$

then:

$$D_{q,yx}f(z_1,z_2) = D_{q,y}\big(D_{q,x}F(z_1,z_2)\big)$$
$$= D_{q,y}g(z_1,z_2)$$
$$= \begin{cases} \dfrac{D_{q,x}f(z_1,z_2)-D_{q,x}f(z_1,qz_2)}{z_2(1-q)}, & z_2 \neq 0 \\ \partial_y D_{q,x}f(z_1,0) \end{cases}$$
$$= \begin{cases} \dfrac{f(z_1,z_2)-f(qz_1,z_2)-f(z_1,qz_2)+f(qz_1,qz_2)}{z_1z_2(1-q)^2}, & z_1 \neq 0, z_2 \neq 0 \\ \dfrac{\partial_x f(0,z_2)-\partial_x f(0,qz_2)}{z_1(1-q)}, & z_1 = 0, z_2 \neq 0 \\ \dfrac{\partial_y f(z_1,0)-\partial_y f(qz_1,0)}{z_1(1-q)}, & z_1 \neq 0, z_2 = 0 \\ \partial_{yx}f(0,0) z_1 = 0, & z_2 = 0. \end{cases}$$

Now let $u = f(z_1,z_2)$, then:

$$D_{q,x}f(z_1,z_2)D_{q,xx}f$$
$$= \begin{cases} \dfrac{\big(f(z_1,z_2)-f(qz_1,z_2)\big)\big(f(z_1,z_2)-2f(qz_1,z_2)+f(q^2z_1,z_2)\big)}{z_1^3(1-q)^3}, & z_1 \neq 0 \\ \partial_x f(0,z_2)\big(\partial_{xx}f(0,z_2)\big), & z_1 = 0. \end{cases} \qquad (4)$$

Now substitute (3) and (4) in (2) we get the following case:

$$\big(D_{q,x}u\big)\big(D_{q,x}^2\big)+\big(D_{q,y}^2\big)$$
$$= \begin{cases} (3)+(4), & z_1 \neq 0 \wedge z_2 \neq 0 \\ \partial_x f(0,z_2)\big(\partial_{xx}f(0,z_2)\big)+(3)\big|_{z_1=0}, & z_1 = 0 \wedge z_2 \neq 0 \\ (4)\big|_{z_2=0}+\partial_{yy}f(0,0), & z_1 \neq 0 \wedge z_2 = 0 \\ \big(\partial_x f(0,0)\big)\big(\partial_{xx}f(0,0)\big)+\partial_{yy}f(0,0), & z_1 = 0 \wedge z_2 = 0. \end{cases}$$

The above discussion gives the following theorem.

Theorem

For $q \in (0,1)$, the solution of the transonic gas Equation 2 is given by:

$$\sum_{i=1}^{r+1}\sum_{j=0}^{s-2}[i]_q[r-i+3]_q[r-i+2]_q$$
$$a_{i,j}a_{r-i+3,s-j-2} = -[s]_q[s-1]_q\, a_{r,s}.$$

Proof

Let *q-transonic* gas equation is $\big(D_{q,x}u\big)\big(D_{q,x}^2\big)+\big(D_{q,y}^2\big) = 0$.

Then, from above discussions, we substitute $D_{q,x}u$, $D_{q,x}^2$ and $D_{q,y}^2$ in (2) we can find:

$$\frac{\left(f(z_1,z_2)-f(qz_1,z_2)\right)\left(f(z_1,z_2)-2f(qz_1,z_2)+f(q^2z_1,z_2)\right)}{z_1^3(1-q)^3}$$

$$+\frac{f(z_1,z_2)-2f(z_1,qz_2)+f(z_1,q^2z_2)}{z_2^2-(1-q)^2}=0,\ \ z_1,z_2\neq 0$$

$$\frac{f^2(z_1,z_2)-3f(z_1,z_2)f(qz_1,z_2)+f(z_1,z_2)f(q^2z_1,z_2)-2f^2(qz_1,z_2)-f(qz_1,z_2)f(q^2z_1,z_2)}{z_1^3(1-q)^3}$$

$$+\frac{f(z_1,z_2)-2f(z_1,qz_2)+f(z_1,q^2z_2)}{z_2^2-(1-q)^2}=0$$

(5)

Now let:

$$u(z_1,z_2)=\sum_{n,m=0}^{\infty}a_{n,m}z_1^n z_2^m,$$

and:

$$D_{q,x}\left(z_1^n\right)=\frac{z_1^n-(qz_1)^n}{z_1(1-q)}$$

$$=\frac{z_1^n\left(1-q^n\right)}{z_1(1-q)}$$

$$=z_1^{n-1}[n]_q$$

This implies:

$$\left(\sum_{n,m}^{\infty}[n]_q a_{n,m}z_1^{n-1}z_2^m\right)\left(\sum_{n,m}^{\infty}[n]_q[n-1]_q a_{n,m}z_1^{n-2}z_2^m\right)$$

$$+\left(\sum_{n,m}^{\infty}[m]_q[m-1]_q a_{n,m}z_1^n z_2^{m-2}\right)=0$$

$$u(z_1,z_2)=\sum_{i,j,n,m}^{\infty}[i]_q a_{i,j}z_1^{i-1}z_2^j[n]_q[n-1]_q a_{n,m}z_1^{n-2}z_2^m$$

$$+\sum_{n,m}^{\infty}[m]_q[m-1]_q a_{n,m}z_1^n z_2^{m-2}=0$$

$$\sum_{i,j,n,m}^{\infty}[i]_q a_{i,j}z_1^{i-1}z_2^j[n]_q[n-1]_q a_{n,m}z_1^{n-2}z_2^m$$

$$+\sum_{n,m}^{\infty}[m]_q[m-1]_q a_{n,m}z_1^n z_1^{m-2}=0$$

This implies:

$$\sum_{i,j,n,m}^{\infty}\left([i]_q[n]_q[n-1]_q a_{i,j}\right)z_1^{i+n-3}z_2^{m+j}$$

$$=\sum_{n,m}^{\infty}-[m]_q[m-1]_q a_{n,m}z_1^n z_2^{m-2},$$

where:

$$\left.\begin{array}{c}m+j=s-2\\i+n-3=r\end{array}\right\}\Rightarrow\begin{array}{c}m=s-j-2,m\geq 0\Leftrightarrow s\geq j+2\\n=r-i+3,n\geq 2\Leftrightarrow r-i+1\geq 0\Rightarrow r\geq i-1\end{array}$$

If and only if:

$$\sum_{i=1,j=0}^{\infty}\sum_{s=j+2,r=i-1}^{\infty}\left([i]_q[r-i+2]_q[r-i+2]_q a_{i,j}a_{r-i+3,s-j-2}\right)z_1^r z_2^{s-2}$$

$$=\sum_{n,m=0}^{\infty}\left(-[m]_q[m-1]_q a_{n,m}\right)z_1^n z_2^{m-2}$$

$$\sum_{r=0,s=2}^{\infty}\left(\sum_{i=1,j=1,}^{r+1}\sum_{}^{s-2}\alpha_{i,j,r,s}\right)z_1^r z_2^{s-2}=\sum_{r=0,\ s=2}^{\infty}\left(-[s]_q[s-1]_q a_{r,s}\right)z_1^r z_2^{s-2},$$

where:

$$\alpha_{i,j,r,s}=\left([i]_q[r-i+2]_q[r-i+2]_q a_{i,j}a_{r-i+3,s-j-2}\right).$$

Hence, we deduce the following result:

$$\sum_{i=1}^{r+1}\sum_{j=0}^{s-2}[i]_q[r-i+3]_q[r-i+2]_q$$

$$a_{i,j}a_{r-i+3,s-j-2}=-[s]_q[s-1]_q a_{r,s}.$$

(6)

$$\forall r\geq 0\ and\ s\geq 2,\forall r\in\{0,1,2,...\}\ and\ s\in\{2,3,4,...\}$$

This conclude the proof.

Conclusion

In this study, the *q-analogous* of transonic gas equation presented. The classical solution considered. We show that the huge different between classical case and a new *q-analogous* of transonic gas equation.

In future I will investigate the solution of Equation (6) using code programming with the software Maple.

Acknowledgement

Ethics

The author declare that there is no conflict interests regarding the publication of this manuscript. This article is original and contains unpublished material.

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