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On The Diophantine Equation $x^a + y^a = p^k z^b$

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Abstract: In this study, we consider the Diophantine equation $x^a + y^a = p^k z^b$ where p is a prime number, $\gcd(a, b) = 1$ and $k,a,b \in \mathbb{Z}^+$. We solve this equation parametrically by considering different cases of x and y and find that there exist infinitely many nontrivial integer solutions, where the formulated parametric solutions solve $x^a + y^a = p^k z^b$ completely for the case of $x = y$, $x = -y$, and either x or y is zero (not both zero). For the case of $|x| \neq |y|$ and both x and y nonzero, not every solution (x,y,z) is in the parametric forms proposed in Theorem 5, although any (x,y,z) in these parametric forms solves the Diophantine equation.

Keywords: Diophantine Equation, Integer Solutions, Congruence, Fundamental Theorem of Arithmetic

Introduction

It is known that there exists no nonzero integer solution to Fermat's equation $x^n + y^n = z^n$ where $n > 2$, as proven by Andrew Wiles in 1995 (Andreescu and Andrica, 2002). However, interests are given to its variations as some may have nonzero integer solutions and there is no universal algorithm that solves any Diophantine equation as proven by Yuri Matiyasevich in 1970 (Steen, 1975). Some examples are $y^3 - x^3 = k$ (Lal *et al*., 1966), $x^2 - Dy^2 = nz^2$ (Cohen, 1992), $x^4 + 2y^4 = z^4 + 4w^4$ (Elsenhans and Jahnel, 2006) and $x^3 + y^3 = z^2$ (Zahari *et al*., 2011).

In 2016, Wong and Kamarulhaili solved the Diophantine equation $x^4 + y^4 = p^k z^7$ (where p is a prime and k is a positive integer) nontrivially in the case of $x = y$, motivated by incomplete parametric solutions proposed by Ismail (2011) for her Diophantine equation of similar form, $x^4 + y^4 = p^k z^3$ (where p is a prime, $p \in [2,13]$ and k is a positive integer). This is due to her assumption in her proofs that z must always contain the prime p when represented as product of primes (Ismail, 2011). Although her parametric solutions can fulfill her Diophantine equation, they would not yield the complete solutions to the equation whenever $k \equiv 1 \pmod 4$ (for $p = 2$) and $k \equiv 0 \pmod 4$ (for $p > 2$) are concerned (Wong, 2016).

Having identified this shortcoming by taking prime factor and congruence consideration into account to solve $x^4 + y^4 = p^k z^7$, we recognized that the same idea can be used on other similar Diophantine equations (Wong and Kamarulhaili, 2016). This motivates the study of:

$$x^a + y^a = p^k z^b \tag{1}$$

where p is a prime number, $\gcd(a,b) = 1$ and $k,a,b \in Z^+$. This paper aims to solve any Diophantine equation in the form of Equation 1. The ideas in proving the theorems in this study are mostly the same as that of Wong and Kamarulhaili (2016), with adjustments made to solve Equation 1 with generalized indices a and b and different cases of x and y.

Our *Main Results* consists of five subsections, where each of them solves the Diophantine equation for different cases of x and y. The first subsection gives the parametric solutions for Equation 1 where $x = y$ and $p = 2$. It considers three cases of k: $k = 1$; $k>1$ and $k \equiv 1 \pmod a$; $k>1$ and $k \not\equiv 1 \pmod a$. The second subsection gives the parametric solutions for Equation 1 where $x = y$ and $p>2$. The third subsection gives the parametric solutions for Equation 1 where $x = -y$ for all p. The fourth subsection gives the parametric solutions for Equation 1 where either x or y is zero (not both zero). The last subsection gives the parametric solutions for Equation 1 where $|x| \neq |y|$ and both x and y nonzero. Second to fifth subsections consider the cases of $k \equiv 0 \pmod a$ and $k \not\equiv 0 \pmod a$.

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Main Results

On the Diophantine Equation $x^a + y^a = p^k z^b$ where $x = y$ and $p = 2$

The following theorem gives the nontrivial parametric solutions to Equation 1 where $x = y$ and $p = 2$.

Theorem 1

Let (x_0, y_0, z_0) be a nontrivial integer solution to $x^a + y^a = 2^k z^b$ where $x_0 = y_0$, $\gcd(a,b) = 1$ and $k, a, b \in \mathbb{Z}^+$. If $k = 1$, then:

$$(x_0, y_0, z_0) = \begin{cases} (\pm n^b, \pm n^b, n^a), & a \text{ even}, b \text{ odd} \\ (n^b, n^b, \pm n^a), & a \text{ odd}, b \text{ even} \\ (\pm n^b, \pm n^b, \pm n^a), & a \text{ and } b \text{ odd} \end{cases} \tag{2}$$

where $n \in \mathbb{Z}^+$. If $k > 1$ and $k \equiv 1 \pmod{a}$, then:

$$(x_0, y_0, z_0) = \begin{cases} (\pm 2^v n^b, \pm 2^v n^b, n^a), & a \text{ even}, b \text{ odd} \\ (2^v n^b, 2^v n^b, \pm n^a), & a \text{ odd}, b \text{ even} \\ (\pm 2^v n^b, \pm 2^v n^b, \pm n^a), & a \text{ and } b \text{ odd} \end{cases} \tag{3}$$

where $n \in \mathbb{Z}^+$ and $av + 1 = k$. If $k > 1$ and $k \not\equiv 1 \pmod{a}$, then:

$$(x_0, y_0, z_0) = \begin{cases} (\pm 2^{c_1(k-1)} n^b, \pm 2^{c_1(k-1)} n^b, 2^{c_2(k-1)} n^a), & a \text{ even}, b \text{ odd} \\ (2^{c_1(k-1)} n^b, 2^{c_1(k-1)} n^b, \pm 2^{c_2(k-1)} n^a), & a \text{ odd}, b \text{ even} \\ (\pm 2^{c_1(k-1)} n^b, \pm 2^{c_1(k-1)} n^b, \pm 2^{c_2(k-1)} n^a), & a \text{ and } b \text{ odd} \end{cases} \tag{4}$$

where $n \in \mathbb{Z}^+$, c_1 and c_2 are smallest nonnegative integers such that $ac_1 - bc_2 = 1$.

Proof

Let (x_0, y_0, z_0) be a solution to $x^a + y^a = 2^k z^b$ where $x_0 = y_0$. Then:

$$x_0^a = 2^{k-1} z_0^b \tag{5}$$

The signs of x_0 and z_0 are influenced by the parities of a and b. For example, when a is odd (respectively, even) and b is even (respectively, odd), x_0 must be positive (respectively, can be positive or negative) and z_0 can be positive or negative (respectively, must be positive). When both a and b are odd, x_0 and z_0 can be either both positive or both negative. a and b cannot be both even since $\gcd(a,b) = 1$. Note that in any possible case, the positives of x_0 and z_0 ensure that Equation 1 always holds. Thus, without loss of generality, let x_0 and z_0 be positive integers. By Fundamental Theorem of Arithmetic, let x_0 and z_0 be represented as a product of primes in their canonical forms, respectively:

$$x_0 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} = \prod_{i=1}^{r} p_i^{\alpha_i} \tag{6}$$

$$z_0 = q_1^{\beta_1} q_2^{\beta_2} \ldots q_s^{\beta_s} = \prod_{j=1}^{s} q_j^{\beta_j} \tag{7}$$

where p_i and q_j are primes, $p_1 < p_2 < \ldots < p_r$, $q_1 < q_2 < \ldots < q_s$, r, s are nonnegative integers and α_i, β_j are positive integers. Plugging Equation 6 and Equation 7 into Equation 5, we get:

$$\prod_{i=1}^{r} p_i^{a\alpha_i} = 2^{k-1} \prod_{j=1}^{s} q_j^{b\beta_j} \tag{8}$$

There are two cases to be considered: $k = 1$; $k > 1$.

Case 1

Suppose $k = 1$. Then Equation 8 becomes:

$$\prod_{i=1}^{r} p_i^{a\alpha_i} = \prod_{j=1}^{s} q_j^{b\beta_j} \tag{9}$$

Due to the uniqueness of canonical representation of integers, we have $r = s$, $p_i = q_j$ and $a\alpha_i = b\beta_j$ for $1 \le i = j \le r = s$. Note that $b \mid a\alpha_i$, $a \mid b\beta_j$ and $\gcd(a,b) = 1$. We must have positive integers v_i and w_j such that $\alpha_i = bv_i$ and $\beta_j = aw_j$. Thus $a(bv_i) = b(aw_j)$ and hence $v_i = w_j$. So, Equation 6 and 7 respectively become:

$$x_0 = y_0 = \prod_{i=1}^{r} p_i^{bv_i} = \left(\prod_{i=1}^{r} p_i^{v_i} \right)^b \tag{10}$$

$$z_0 = \prod_{j=1}^{s} q_j^{aw_j} = \left(\prod_{j=1}^{s} q_j^{w_j} \right)^a \tag{11}$$

Let $n = \prod_{i=1}^{r} p_i^{v_i} = \prod_{j=1}^{s} q_j^{w_j}$. Then from Equation 10 and 11, we have $x_0 = y_0 = n^b$ and $z_0 = n^a$. Note that n is any positive integer. Depending on the parities of a and b, (x_0, y_0, z_0) is in one of the following forms:

$$(x_0, y_0, z_0) = \begin{cases} (\pm n^b, \pm n^b, n^a), & a \text{ even}, b \text{ odd} \\ (n^b, n^b, \pm n^a), & a \text{ odd}, b \text{ even} \\ (\pm n^b, \pm n^b, \pm n^a), & a \text{ and } b \text{ odd} \end{cases} \tag{12}$$

where, $n \in \mathbb{Z}^{+}$.

Case 2

Suppose $k>1$. Then from Equation 5, we know that $2^{k-1} \mid x_0^a$. Thus, $x_0 = \prod_{i=1}^{r} p_i^{\alpha_i}$ must have a prime factor $p_1 = 2$. Equation 6 becomes:

$$x_0 = 2^{\alpha_1} \prod_{i=2}^{r} p_i^{\alpha_i} \tag{13}$$

and Equation 8 becomes:

$$2^{a\alpha_1} \prod_{i=2}^{r} p_i^{a\alpha_i} = 2^{k-1} \prod_{j=1}^{s} q_j^{b\beta_j} \tag{14}$$

As we compare the indices of 2 on both sides of Equation 14, $a\alpha_1 = k-1$ suggests that only positive integers k where $k \equiv 1$ (mod a) allow Equation 14 to be consistent. However, in the case of $k \not\equiv 1$ (mod a), Equation 14 still holds if z_0 has a prime factor 2. Hence, there are two cases that need to be considered: $k \not\equiv 1$ (mod a); $k \equiv 1$ (mod a).

Case a

Suppose $k \not\equiv 1$ (mod a). Equation 14 will not be consistent unless there exists a prime factor $q_1 = 2$ in z_0. Thus $z_0 = 2^{\beta_1} \prod_{j=2}^{s} q_j^{\beta_j}$ and Equation 14 becomes:

$$2^{a\alpha_1} \prod_{i=2}^{r} p_i^{a\alpha_i} = 2^{k-1+b\beta_1} \prod_{j=2}^{s} q_j^{b\beta_j} \tag{15}$$

Compare the indices of 2 on both sides of Equation 15. It is easy to check that for the linear Diophantine problem $a\alpha_1 - b\beta_1 = k-1$, $(\alpha_1, \beta_1) = (c_1(k-1), c_2(k-1))$ is a particular solution where c_1 and c_2 are smallest nonnegative integers such that $ac_1 - bc_2 = 1$. Its general solution is thus $\alpha_1 = c_1(k-1) - bt$ and $\beta_1 = c_2(k-1) - at$ where $t \in \mathbb{Z}_{\leq 0}$ ($\mathbb{Z}_{\leq 0}$ refers to set of integers less than or equal to 0). Any integer represented in its canonical form is unique. Thus $r = s$, $p_i = q_j$ and $a\alpha_i = b\beta_j$ for $2 \leq i = j \leq r = s$. Note that $b \mid a\alpha_i$, $a \mid b\beta_j$ and $\gcd(a,b) = 1$. We must have positive integers v_i and w_j such that $\alpha_i = bv_i$ and $\beta_j = aw_j$. Thus $a(bv_i) = b(aw_j)$ and hence $v_i = w_j$. Using these new forms, we have:

$$x_0 = y_0 = 2^{c_1(k-1)} \left(2^{-t} \prod_{i=2}^{r} p_i^{v_i} \right)^b \tag{16}$$

$$z_0 = 2^{c_2(k-1)} \left(2^{-t} \prod_{j=2}^{s} q_j^{w_j} \right)^a \tag{17}$$

Let $n = 2^{-t} \prod_{i=2}^{r} p_i^{v_i} = 2^{-t} \prod_{j=2}^{s} q_j^{w_j}$. Then Equation 16 and 17 become $x_0 = y_0 = 2^{c_1(k-1)} n^b$ and $z_0 = 2^{c_2(k-1)} n^a$, respectively. Depending on the parities of a and b, (x_0, y_0, z_0) is in one of the following forms:

$$(x_0, y_0, z_0) = \begin{cases} (\pm 2^{c_1(k-1)} n^b, \pm 2^{c_1(k-1)} n^b, 2^{c_2(k-1)} n^a), & a \text{ even}, b \text{ odd} \\ (2^{c_1(k-1)} n^b, 2^{c_1(k-1)} n^b, \pm 2^{c_2(k-1)} n^a), & a \text{ odd}, b \text{ even} \\ (\pm 2^{c_1(k-1)} n^b, \pm 2^{c_1(k-1)} n^b, \pm 2^{c_2(k-1)} n^a), & a \text{ and } b \text{ odd} \end{cases} \tag{18}$$

where $n \in \mathbb{Z}^{+}$.

Case b

Suppose $k \equiv 1$ (mod a). Then there are another two subcases to be considered: z_0 has a prime factor $q_1 = 2$; z_0 does not have a prime factor 2.

Case b1

Suppose z_0 has a prime factor $q_1 = 2$. Then we have $z_0 = 2^{\beta_1} \prod_{j=2}^{s} q_j^{\beta_j}$ and Equation 14 becomes Equation 15. This subcase is solved using the same steps in *Case a*, yielding the same forms of (x_0, y_0, z_0) in the said case.

Case b2

Suppose z_0 does not have a prime factor 2. Then z_0 remains as Equation 7 and in order for Equation 14 to be consistent, $s = r-1$. Thus Equation 14 becomes:

$$2^{a\alpha_1} \prod_{i=2}^{r} p_i^{a\alpha_i} = 2^{k-1} \prod_{j=2}^{r} q_j^{b\beta_j} \tag{19}$$

Compare the indices of 2 on both sides of Equation 19 and we see that $a\alpha_1 = k-1$. Since $a \mid (k-1)$, there exists a positive integer v such that $k = av+1$. Thus, $a\alpha_1 = (av+1) - 1$ which leads to $\alpha_1 = v$. Again, due to the uniqueness of canonical form of integers, $p_i = q_j$ for $2 \leq i = j \leq r$. $b \mid a\alpha_i$, $a \mid b\beta_j$ and $\gcd(a,b) = 1$. We must have positive integers v_i and w_j such that $\alpha_i = bv_i$ and $\beta_j = aw_j$. This leads to $v_i = w_j$. Using these new forms, we have:

$$x_0 = y_0 = 2^{v} \left(\prod_{i=2}^{r} p_i^{v_i} \right)^b \tag{20}$$

$$z_0 = \left(\prod_{j=2}^{r} q_j^{w_j}\right)^a \tag{21}$$

Let $n = \prod_{i=2}^{r} p_i^{v_i} = \prod_{j=2}^{r} q_j^{w_j}$ and thus Equation 20 and 21 become $x_0 = y_0 = 2^v n^b$ and $z_0 = n^a$, respectively. Depending on the parities of a and b, (x_0, y_0, z_0) is in one of the following forms:

$$(x_0, y_0, z_0) = \begin{cases} (\pm 2^v n^b, \pm 2^v n^b, n^a), \; a \text{ even}, b \text{ odd} \\ (2^v n^b, 2^v n^b, \pm n^a), \quad a \text{ odd}, b \text{ even} \\ (\pm 2^v n^b, \pm 2^v n^b, \pm n^a), \; a \text{ and } b \text{ odd} \end{cases} \tag{22}$$

where $n \in \mathbb{Z}^+$. This form of solution covers that of *Case b1* where z_0 has a prime factor 2 and $k \equiv 1 \pmod{a}$. It is easy to show that for a choice of $n = n_1$ in *Case b1*, the same solution can be obtained at $n = 2^{c_2 v} n_1$ in *Case b2*. Thus, in the case of $k \equiv 1 \pmod{a}$, the parametric forms in *Case b2* is sufficient to solve for all (x_0, y_0, z_0).

On the Diophantine Equation $x^a + y^a = p^k z^b$ where $x = y$ and $p > 2$

The following theorem gives the nontrivial parametric solutions to Equation 1 where $x = y$ and $p>2$.

Theorem 2

Let (x_0, y_0, z_0) be a nontrivial integer solution to $x^a + y^a = p^k z^b$ where $x_0 = y_0$, $\gcd(a,b) = 1$, $k, a, b \in \mathbb{Z}^+$ and p is a prime number where $p > 2$. If $k \equiv 0 \pmod{a}$, then:

$$(x_0, y_0, z_0) = \begin{cases} (\pm 2^{c_1} p^v n^b, \pm 2^{c_1} p^v n^b, 2^{c_2} n^a), \; a \text{ even}, b \text{ odd} \\ (2^{c_1} p^v n^b, 2^{c_1} p^v n^b, \pm 2^{c_2} n^a), \quad a \text{ odd}, b \text{ even} \\ (\pm 2^{c_1} p^v n^b, \pm 2^{c_1} p^v n^b, \pm 2^{c_2} n^a), \; a \text{ and } b \text{ odd} \end{cases} \tag{23}$$

where $n \in \mathbb{Z}^+$, $av = k$ and c_1 and c_2 are smallest nonnegative integers such that $-ac_1 + bc_2 = 1$. If $k \not\equiv 0 \pmod{a}$, then:

$$(x_0, y_0, z_0) = \begin{cases} (\pm 2^{c_1} p^{d_1 k} n^b, \pm 2^{c_1} p^{d_1 k} n^b, 2^{c_2} p^{d_2 k} n^a), \; a \text{ even}, b \text{ odd} \\ (2^{c_1} p^{d_1 k} n^b, 2^{c_1} p^{d_1 k} n^b, \pm 2^{c_2} p^{d_2 k} n^a), \quad a \text{ odd}, b \text{ even} \\ (\pm 2^{c_1} p^{d_1 k} n^b, \pm 2^{c_1} p^{d_1 k} n^b, \pm 2^{c_2} p^{d_2 k} n^a), \; a \text{ and } b \text{ odd} \end{cases} \tag{24}$$

where $n \in \mathbb{Z}^+$ and c_1, c_2, d_1 and d_2 are smallest nonnegative integers such that $-ac_1 + bc_2 = 1$ and $ad_1 - bd_2 = 1$.

Proof

Let (x_0, y_0, z_0) be a solution to Equation 1 where $x_0 = y_0$. Then:

$$2x_0^a = p^k z_0^b \tag{25}$$

Without loss of generality, let x_0 and z_0 be positive integers. By Fundamental Theorem of Arithmetic, let x_0 and z_0 be represented as a product of primes $\prod_{i=1}^{r} p_i^{\alpha_i}$ and $\prod_{j=1}^{s} q_j^{\beta_j}$, respectively, where p_i and q_j are primes, r and s are nonnegative integers and α_i and β_j are positive integers. Since $\gcd(2, p^k) = 1$, we know that $2 \mid z_0^b$ and $p^k \mid x_0^a$. There must be prime factors $p_i = p$ in x_0 and $q_j = 2$ in z_0, where $1 \le i \le r$ and $1 \le j \le s$. Let $p_2 = p$ and $q_1 = 2$. Then:

$$x_0 = p^{\alpha_2} \prod_{i=1, i \ne 2}^{r} p_i^{\alpha_i} \tag{26}$$

$$z_0 = 2^{\beta_1} \prod_{j=2}^{s} q_j^{\beta_j} \tag{27}$$

Plugging Equation 26 and 27 into Equation 25, we get:

$$2p^{a\alpha_2} \prod_{i=1, i \ne 2}^{r} p_i^{a\alpha_i} = 2^{b\beta_1} p^k \prod_{j=2}^{s} q_j^{b\beta_j} \tag{28}$$

Observe the indices of 2 in Equation 28 that in order for Equation 28 to be consistent, $b\beta_1 = 1$ but $\beta_1 \notin \mathbb{Z}^+$ (contradiction). Thus, x_0 must contain a prime factor $p_i = 2$ where $1 \le i \le r$, $i \ne 2$. Let $p_1 = 2$ and we have $x_0 = 2^{\alpha_1} p^{\alpha_2} \prod_{i=3}^{r} p_i^{\alpha_i}$. Then Equation 28 becomes:

$$2^{a\alpha_1 + 1} p^{a\alpha_2} \prod_{i=3}^{r} p_i^{a\alpha_i} = 2^{b\beta_1} p^k \prod_{j=2}^{s} q_j^{b\beta_j} \tag{29}$$

As for the indices of p in Equation 29, $a\alpha_2 = k$ suggests that only positive integers k where $k \equiv 0 \pmod{a}$ allow Equation 29 to be consistent. However, in the case of $k \not\equiv 0 \pmod{a}$, Equation 29 still holds if z_0 has a prime factor p. Hence, there are two cases that need to be considered: $k \not\equiv 0 \pmod{a}$; $k \equiv 0 \pmod{a}$.

Case a

Suppose $k \not\equiv 0 \pmod{a}$. Then Equation 29 will not be consistent unless there exist a prime $q_j = p$ where $2 \le j \le s$. Let $q_2 = p$ and we have $z_0 = 2^{\beta_1} p^{\beta_2} \prod_{j=3}^{s} q_j^{\beta_j}$. Then Equation 29 becomes:

$$2^{a\alpha_1+1} p^{a\alpha_2} \prod_{i=3}^{r} p_i^{a\alpha_i} = 2^{b\beta_1} p^{b\beta_2+k} \prod_{j=3}^{s} q_j^{b\beta_j} \quad (30)$$

Compare the indices of 2 and p on both sides of Equation 30. There are two linear Diophantine problems to be solved in order for Equation 30 to be consistent:

$$-a\alpha_1 + b\beta_1 = 1 \quad (31)$$

$$a\alpha_2 - b\beta_2 = k \quad (32)$$

For Equation 31, its solution is $\alpha_1 = c_1 - bt_1$ and $\beta_1 = c_2 - at_1$ where c_1 and c_2 are smallest nonnegative integers such that $-ac_1 + bc_2 = 1$ and $t_1 \in \mathbb{Z}_{\le 0}$. For Equation 32, its solution is $\alpha_2 = d_1 k - bt_2$ and $\beta_2 = d_2 k - at_2$ where d_1 and d_2 are smallest nonnegative integers such that $ad_1 - bd_2 = 1$ and $t_2 \in \mathbb{Z}_{\le 0}$.

Let σ and λ be permutation functions defined respectively as:

$$\sigma : \{3,4,...,r\} \mapsto \{3,4,...,r\} \quad (33)$$

where $\sigma(i) = i'$, $p_{i'} < p_{i'+1}$, $i' < i'+1$ and:

$$\lambda : \{3,4,...,s\} \mapsto \{3,4,...,s\} \quad (34)$$

where $\lambda(j) = j'$, $q_{j'} < q_{j'+1}$, $j' < j'+1$.

We apply them on the products of primes in Equation 30 to arrange the primes in $\prod_{i=3}^{r} p_i^{a\alpha_i}$ and $\prod_{j=3}^{s} q_j^{b\beta_j}$ into their canonical forms and Equation 30 becomes:

$$2^{a\alpha_1+1} p^{a\alpha_2} \prod_{i'=3}^{r} p_{i'}^{a\alpha_{i'}} = 2^{b\beta_1} p^{b\beta_2+k} \prod_{j'=3}^{s} q_{j'}^{b\beta_{j'}} \quad (35)$$

Any integer represented in its canonical form is unique. So we have $r = s$ and $p_{i'} = q_{j'}$ for $3 \le i' = j' \le r = s$ for Equation 35 to be consistent. Consequently, $a\alpha_{i'} = b\beta_{j'}$. $b \mid a\alpha_{i'}$, $a \mid b\beta_{j'}$ and $\gcd(a,b) = 1$. We must have positive integers $v_{i'}$ and $w_{j'}$ such that $\alpha_{i'} = bv_{i'}$ and $\beta_{j'} = aw_{j'}$. Thus, $a(bv_{i'}) = b(aw_{j'})$ and hence $v_{i'} = w_{j'}$. Using these new forms, we have:

$$x_0 = y_0 = 2^{c_1} p^{d_1 k} \left(2^{-t_1} p^{-t_2} \prod_{i'=3}^{r} p_{i'}^{v_{i'}} \right)^b \quad (36)$$

$$z_0 = 2^{c_2} p^{d_2 k} \left(2^{-t_1} p^{-t_2} \prod_{j'=3}^{s} q_{j'}^{w_{j'}} \right)^a \quad (37)$$

Let $n = 2^{-t_1} p^{-t_2} \prod_{i'=3}^{r} p_{i'}^{v_{i'}} = 2^{-t_1} p^{-t_2} \prod_{j'=3}^{s} q_{j'}^{w_{j'}}$ and thus Equation 36 and 37 become $x_0 = y_0 = 2^{c_1} p^{d_1 k} n^b$ and $z_0 = 2^{c_2} p^{d_2 k} n^a$, respectively. Thus, in the case of $k \not\equiv 0$ (mod a), depending on the parities of a and b, (x_0, y_0, z_0) is in one of the following forms:

$$(x_0,y_0,z_0) = \begin{cases} (\pm 2^{c_1} p^{d_1 k} n^b, \pm 2^{c_1} p^{d_1 k} n^b, 2^{c_2} p^{d_2 k} n^a), & a \text{ even}, b \text{ odd} \\ (2^{c_1} p^{d_1 k} n^b, 2^{c_1} p^{d_1 k} n^b, \pm 2^{c_2} p^{d_2 k} n^a), & a \text{ odd}, b \text{ even} \\ (\pm 2^{c_1} p^{d_1 k} n^b, \pm 2^{c_1} p^{d_1 k} n^b, \pm 2^{c_2} p^{d_2 k} n^a), & a \text{ and } b \text{ odd} \end{cases} \quad (38)$$

where $n \in \mathbb{Z}^+$.

Case b

Suppose that $k \equiv 0$ (mod a). Then there are another two subcases to be considered: z_0 has a prime factor $q_j = p$ where $2 \le j \le s$; z_0 does not have a prime factor p (refer to Equation 29).

Case b1

Suppose that z_0 has a prime factor $q_j = p$ where $2 \le j \le s$. Let $q_2 = p$ and we have $z_0 = 2^{\beta_1} p^{\beta_2} \prod_{j=3}^{s} q_j^{\beta_j}$ and Equation 29 becomes Equation 30. This subcase is solved using the same steps in *Case a*, yielding the same forms of (x_0, y_0, z_0) in the said case.

Case b2

Suppose that z_0 does not have a prime factor p. Then z_0 is in the form of Equation 27 and in order for Equation 29 to be consistent, $s = r - 1$. Thus Equation 29 becomes:

$$2^{a\alpha_1+1} p^{a\alpha_2} \prod_{i=3}^{r} p_i^{a\alpha_i} = 2^{b\beta_1} p^{k} \prod_{j=3}^{r} q_j^{b\beta_j} \quad (39)$$

By comparing the indices of 2 and p on both sides of Equation 39, we have two equations to solve:

$$-a\alpha_1 + b\beta_1 = 1 \quad (40)$$

$$a\alpha_2 = k \quad (41)$$

Equation 40 is the same as Equation 31 in *Case a* with the solution of $\alpha_1 = c_1 - bt$ and $\beta_1 = c_2 - at$ where c_1 and c_2 are smallest nonnegative integers such that $-ac_1 + bc_2 = 1$ and $t \in \mathbb{Z}_{\le 0}$. For Equation 41, note that $a|k$, so there exists a positive integer v such that $k = av$. So $a\alpha_2 = av$ and hence $\alpha_2 = v$. Apply Equation 33 and 34 (where the domain and codomain of Equation 34 are now $\{3,4,\ldots r\}$) on the products of primes in Equation 39 to arrange them into canonical forms and we get:

$$2^{a\alpha_1+1}p^{a\alpha_2}\prod_{i'=3}^{r}p_{i'}^{a\alpha_{i'}}=2^{b\beta_1}p^k\prod_{j'=3}^{r}q_{j'}^{b\beta_{j'}} \tag{42}$$

Any integer represented in its canonical form is unique, so $p_{i'}=q_{j'}$ and $a\alpha_{i'}=b\beta_{j'}$ for $3\le i'=j'\le r$. $b\mid a\alpha_{i'}$, $a\mid b\beta_{j'}$ and $\gcd(a,b)=1$. We must have positive integers $v_{i'}$ and $w_{j'}$ such that $\alpha_{i'}=bv_{i'}$ and $\beta_{j'}=aw_{j'}$. So, $a(bv_{i'})=b(aw_{j'})$ and hence $v_{i'}=w_{j'}$. Using these new forms, we have:

$$x_0=y_0=2^{c_1}p^v\left(2^{-t}\prod_{i'=3}^{r}p_{i'}^{v_{i'}}\right)^b \tag{43}$$

$$z_0=2^{c_2}\left(2^{-t}\prod_{j'=3}^{r}q_{j'}^{w_{j'}}\right)^a \tag{44}$$

Let $n=2^{-t}\prod_{i'=3}^{r}p_{i'}^{v_{i'}}=2^{-t}\prod_{j'=3}^{r}q_{j'}^{w_{j'}}$ and thus Equation 43 and 44 become $x_0=y_0=2^{c_1}p^vn^b$ and $z_0=2^{c_2}n^a$, respectively. Thus, in the case of $k\equiv 0 \pmod a$, depending on the parities of a and b, (x_0,y_0,z_0) is in one of the following forms:

$$(x_0,y_0,z_0)=\begin{cases}(\pm 2^{c_1}p^vn^b,\pm 2^{c_1}p^vn^b,2^{c_2}n^a), & a \text{ even}, b \text{ odd}\\ (2^{c_1}p^vn^b,2^{c_1}p^vn^b,\pm 2^{c_2}n^a), & a \text{ odd}, b \text{ even}\\ (\pm 2^{c_1}p^vn^b,\pm 2^{c_1}p^vn^b,\pm 2^{c_2}n^a), & a \text{ and } b \text{ odd}\end{cases} \tag{45}$$

where $n\in\mathbb{Z}^+$. This form of solution covers that of *Case b1* where z_0 has a prime factor p and $k\equiv 0 \pmod a$. It is easy to show that for a choice of $n=n_1$ in *Case b1*, the same solution can be obtained at $n=p^{d_2v}n_1$ in *Case b2*. Thus, in the case of $k\equiv 0 \pmod a$, the parametric forms in *Case b2* is sufficient to solve for all (x_0,y_0,z_0).

On the Diophantine Equation $x^a + y^a = p^kz^b$ where $x = -y$

The following corollary gives the nontrivial parametric solutions to Equation 1 where $x=-y$.

Corollary 3

Let (x_0,y_0,z_0) be a nontrivial integer solution to $x^a+y^a=p^kz^b$ where $x_0=-y_0$, $\gcd(a,b)=1$, $k,a,b\in\mathbb{Z}^+$ and p is a prime number. If $p=2$ and a even, then:

$$(x_0,y_0,z_0)=\begin{cases}(\pm n^b,\mp n^b,n^a), & k=1\\ (\pm 2^vn^b,\mp 2^vn^b,n^a), & k>1, k\equiv 1 \pmod a\\ (\pm 2^{c_1(k-1)}n^b,\mp 2^{c_1(k-1)}n^b,2^{c_2(k-1)}n^a), & k>1, k\not\equiv 1 \pmod a\end{cases} \tag{46}$$

where, $n\in\mathbb{Z}^+$, $av+1=k$ and c_1 and c_2 are smallest nonnegative integers such that $-ac_1+bc_2=1$. If $p>2$ and a is even, then:

$$(x_0,y_0,z_0)=\begin{cases}(\pm 2^{c_1}p^vn^b,\mp 2^{c_1}p^vn^b,2^{c_2}n^a), & k\equiv 0 \pmod a\\ (\pm 2^{c_1}p^{d_1k}n^b,\mp 2^{c_1}p^{d_1k}n^b,2^{c_2}p^{d_2k}n^a), & k\not\equiv 0 \pmod a\end{cases} \tag{47}$$

where, $n\in\mathbb{Z}^+$, $av=k$ and c_1, c_2, d_1 and d_2 are smallest nonnegative integers such that $-ac_1+bc_2=1$ and $ad_1-bd_2=1$. If a is odd, then:

$$(x_0,y_0,z_0)=(\pm n,\mp n,0) \tag{48}$$

where $n\in\mathbb{Z}^+$.

Proof

Let (x_0,y_0,z_0) be a solution to Equation 1 where $x_0=-y_0$. Then:

$$x_0^a+(-x_0)^a=p^kz_0^b \tag{49}$$

When a is odd, clearly this results in $z_0=0$ for any x_0 and thus we have $(x_0,y_0,z_0)=(\pm n,\mp n,0)$ where $n\in\mathbb{Z}^+$. When a is even (and b is odd), $x_0^a+(-x_0)^a$ $=x_0^a+(-1)^ax_0^a=2x_0^a$ and Equation 49 becomes:

$$2x_0^a=p^kz_0^b \tag{50}$$

which is the same as Equation 5 (where $p = 2$) and Equation 25 (where $p>2$). The steps in finding the parametric solutions are thus the same as those in Theorems 1 and 2, leading to the same forms of solution with the exception that the signs of x_0 and y_0 are opposite in the end. Hence we have the parametric solutions as asserted.

On the Diophantine Equation $x^a + y^a = p^kz^b$ where x or y is Zero (Not Both Zero)

The following theorem gives the nontrivial parametric solutions to Equation 1 where x or y is zero (not both zero).

Theorem 4

Let (x_0,y_0,z_0) be a nontrivial integer solution to $x^a+y^a=p^kz^b$ where either x_0 or y_0 is zero (not both zero), $\gcd(a,b)=1$, $k,a,b\in\mathbb{Z}^+$ and p is a prime number. If $k\equiv 0$ (mod a), then:

$$(x_0,y_0,z_0)=\begin{cases}(\pm p^vn^b,0,n^a)\text{or}(0,\pm p^vn^b,n^a), & a \text{ even}, b \text{ odd}\\ (p^vn^b,0,\pm n^a)\text{or}(0,p^vn^b,\pm n^a), & a \text{ odd}, b \text{ even}\\ (\pm p^vn^b,0,\pm n^a)\text{or}(0,\pm p^vn^b,\pm n^a), & a \text{ and } b \text{ odd}\end{cases} \tag{51}$$

where, $n \in \mathbb{Z}^+$ and $av = k$. If $k \not\equiv 0 \pmod a$, then:

$$(x_0, y_0, z_0) = \begin{cases} (\pm p^{c_1 k} n^b, 0, p^{c_2 k} n^a) \text{or} (0, \pm p^{c_1 k} n^b, p^{c_2 k} n^a), & a \text{ even}, b \text{ odd} \\ (p^{c_1 k} n^b, 0, \pm p^{c_2 k} n^a) \text{or} (0, p^{c_1 k} n^b, \pm p^{c_2 k} n^a), & a \text{ odd}, b \text{ even} \\ (\pm p^{c_1 k} n^b, 0, \pm p^{c_2 k} n^a) \text{or} (0, \pm p^{c_1 k} n^b, \pm p^{c_2 k} n^a), & a \text{ and } b \text{ odd} \end{cases} \quad (52)$$

where $n \in \mathbb{Z}^+$ and c_1 and c_2 are smallest nonnegative integers such that $ac_1 - bc_2 = 1$.

Proof

Similar idea of using the Fundamental Theorem of Arithmetic, prime factor consideration and congruence of k from Theorem 2 is applied, with one of the pair (x_0, y_0) now being zero.

On the Diophantine Equation $x^a + y^a = p^k z^b$ where $|x| \neq |y|$ and Both x and y Nonzero

The following theorem gives some nontrivial parametric solutions to Equation 1 where $|x| \neq |y|$ and both x and y nonzero.

Theorem 5

If $(x_0, y_0, z_0) = (p^v c_1 n^{g_1}, p^v c_2 n^{g_1}, n^{g_2})$ where $k \equiv 0 \pmod a$, $av = k$, c_1 and c_2 are nonzero integers, $|c_1| \neq |c_2|$, $n = c_1^a + c_2^a$, g_1 and g_2 are smallest nonnegative integers such that $-ag_1 + bg_2 = 1$, or $(x_0, y_0, z_0) = \left(p^{f_1 k} c_1 n^{g_1}, p^{f_1 k} c_2 n^{g_1}, p^{f_2 k} n^{g_2}\right)$ where $k \not\equiv 0 \pmod a$, c_1 and c_2 are nonzero integers, $|c_1| \neq |c_2|$, $n = c_1^a + c_2^a$ and f_1, f_2, g_1 and g_2 are smallest nonnegative integers such that $af_1 - bf_2 = 1$ and $-ag_1 + bg_2 = 1$, then (x_0, y_0, z_0) is a nontrivial integer solution to $x^a + y^a = p^k z^b$ where $|x_0| \neq |y_0|$, both x_0 and y_0 nonzero, $\gcd(a,b) = 1$, $k, a, b \in \mathbb{Z}^+$ and p is a prime number.

Proof

It is easy to show that the parametric forms above fulfill Equation 1. The idea of using two independent variables c_1 and c_2 to balance the equation for such case comes from Ismail (2011), although she did not consider the congruence of k which leads to missing out on some solutions. The following is the process through which the proposed parametric forms came to be.

Let (x_0, y_0, z_0) be a solution to Equation 1 where $x_0 = p^\alpha c_1 n^\beta$, $y_0 = p^\alpha c_2 n^\beta$, p is a prime number, c_1, c_2 and n are nonzero integers, $|c_1| \neq |c_2|$ and $\alpha, \beta \in \mathbb{Z}^+$. Plugging them into Equation 1, we get:

$$p^{a\alpha} c_1^a n^{a\beta} + p^{a\alpha} c_2^a n^{a\beta} = p^k z_0^b \quad (53)$$

and thus:

$$p^{a\alpha} n^{a\beta} (c_1^a + c_2^a) = p^k z_0^b \quad (54)$$

Setting $n = c_1^a + c_2^a$, we get:

$$p^{a\alpha} n^{a\beta + 1} = p^k z_0^b \quad (55)$$

We shall consider the congruence of k: $k \not\equiv 0 \pmod a$; $k \equiv 0 \pmod a$.

Case a

Suppose that $k \not\equiv 0 \pmod a$. z_0 must have factors p and n for Equation 55 to be consistent. Let $z_0 = p^\gamma n^\theta$. Then:

$$p^{a\alpha} n^{a\beta + 1} = p^{k + b\gamma} n^{b\theta} \quad (56)$$

Compare the indices of p and n on both sides of Equation 56. There are two linear Diophantine problems to be solved:

$$a\alpha - b\gamma = k \quad (57)$$

$$-a\beta + b\theta = 1 \quad (58)$$

For Equation 57, the general solution is $\alpha = f_1 k - bt_1$ and $\gamma = f_2 k - at_1$ where f_1 and f_2 are smallest nonnegative integers such that $af_1 - bf_2 = 1$ and $t_1 \in \mathbb{Z}_{\leq 0}$. For Equation 58, the general solution is $\beta = g_1 - bt_2$ and $\theta = g_2 - at_2$ where g_1 and g_2 are smallest nonnegative integers such that $-ag_1 + bg_2 = 1$ and $t_2 \in \mathbb{Z}_{\leq 0}$. Setting $t_1 = t_2 = 0$, we have the parametric form of $(x_0, y_0, z_0) = \left(p^{f_1 k} c_1 n^{g_1}, p^{f_1 k} c_2 n^{g_1}, p^{f_2 k} n^{g_2}\right)$ as asserted.

Case b

Suppose that $k \equiv 0 \pmod a$. z_0 must have a factor n for Equation 55 to be consistent. Let $z_0 = n^\theta$. Then:

$$p^{a\alpha} n^{a\beta + 1} = p^k n^{b\theta} \quad (59)$$

Compare the indices of p and n on both sides of Equation 59. There are two equations to be solved:

$$a\alpha = k \quad (60)$$

$$-a\beta + b\theta = 1 \quad (61)$$

For Equation 60, we see that $a \mid k$ so there exists a positive integer v such that $av = k$. Then $a\alpha = av$ and thus $\alpha = v$. For Equation 61 which is the same as

Equation 58, the general solution is $\beta = g_1 - bt$ and $\theta = g_2 - at$ where g_1 and g_2 are smallest nonnegative integers such that $-ag_1 + bg_2 = 1$ and $t \in \mathbb{Z}_{\leq 0}$. Setting t = 0, we have $(x_0, y_0, z_0) = (p^v c_1 n^{g_1}, p^v c_2 n^{g_1}, n^{g_2})$ as asserted.

Note that these proposed parametric solutions do not cover all solutions for Equation 1 for the case of $|x| \neq |y|$. For example, it could not yield $(x, y, z) = (1, 2, 1)$ for $x^4 + y^4 = 17z^7$. However, any (x, y, z) in the parametric forms stated in Theorem 5 is a solution for Equation 1 for the case of $|x| \neq |y|$ and both x and y nonzero. For example, $(x, y, z) =$ (410338673, 820677346, 83521) is a solution to $x^4 + y^4 = 17z^7$ that can be found using Theorem 5.

Conclusion

In this study, we considered different cases of (x, y) to find the parametric solutions to Equation 1. For $x = y$, Theorems 1 and 2 solve Equation 1 completely. For $x = -y$, Corollary 3 solves Equation 1 completely. For either x or y is zero (not both zero), Theorem 4 solves Equation 1 completely. For $|x| \neq |y|$ and both x and y nonzero, any (x, y, z) in the parametric forms stated in Theorem 5 is a solution to Equation 1.

From these results, we know that there exist infinitely many nontrivial integer solutions to Equation 1. The parametric solutions formulated in the main results enable quick and easy solution generation whenever a Diophantine equation in such form is encountered. For example, $x^5 + y^5 = 23^{14}z^7$, $x^{11} + y^{11} = 3^{10}z^6$ and $x^7 + y^7 = 11z^3$. The ideas in this study can also be used for other Diophantine equations that are of similar forms.

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Authors' Contributions

Keng Yarn Wong: Initial preparation, development and publication of this manuscript.

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Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issue is expected after the publication of this manuscript.

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