NEW CONSTRUCTION METHOD OF RECTANGULAR PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS AND SINGULAR GROUP DIVISIBLE DESIGNS

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ABSTRACT

In this study, we propose a new method of construction of rectangular Partially Balanced Incomplete Block (PBIB) designs of size k = 2s and $2 \leq s < l$ and singular Group Divisible (GD) designs of size k = 2l, l is the number of columns in a rectangular matrix containing $v = n \times l$ treatments. We give the parameters expression of the obtained designs. Furthermore, we provide two algorithms of calculation for practical use.

Keywords: Rectangular Association Scheme, Group Divisible Association Scheme, Partially Balanced Incomplete Block Design

1. INTRODUCTION

A PBIB designs, based on an m-association scheme, with parameters v, b, r, k, λ_i, i = 1,2,3...m, is a block design with v treatments and b blocks of size k each such that every treatment occurs in r blocks and any two distinct treatments being i^{th} associate occur together in exactly λ_i blocks. For a PBIB design based on an m-class association scheme with parameters v, b, r, k, n_i it is shown by (Bose and Nair, 1938), that Equation (1 and 2):

$$vr = bk \tag{1}$$

$$\sum_{i=1}^{m} \lambda_i n_i = r(k-1) \tag{2}$$

Rectangular designs, introduced by (Vartak, 1955), are 3-associated PBIB designs based on rectangular association scheme of $v = n \times l$ treatments arranged in $n \times l$ rectangular array such that, with respect to each treatment, the first associates are the other l-1 (= n_1 say) treatments of the same row, the second associates are other n-1 (= n_2 say) treatments of the same column and the remaining (n-1) (l-1) (= n_3 say) treatments are the third associates (Bailey, 2004).

A Group Divisible (GD) design is a 2-associated PBIB design based on a group divisible association scheme, i.e., a set of $v = n \times l$ treatments can be divided into n groups of l treatments each such that any two treatments occur together in λ_1 blocks if they belong to the same group and in λ_2 blocks if they belong to different groups (Bailey, 2004).

The experimental rectangular designs are important subclasses of partially balanced block designs. They were introduced for the first time by (Vartak, 1955), since then, several construction methods have been proposed by (Singh *et al.*, 2011; Kageyama and Sinha, 2003) using essentially the incidence matrices.

In our work, we propose an original method of construction from a rectangular association schemes. Block designs obtained are of size k = 2s and $2 \leq s < l$. The special case s = l, provides singular group divisible designs.

2. DESRIPTION OF THE COMBINATORY METHOD (S)

Let v = nl treatments arranged in an array of n rows and l columns as follows:

a_{11}	a_{12}	...	a_{1j}	...	a_{1l}
a_{21}	a_{22}	...	a_{2j}	...	a_{2l}
...	...	...	...	...	...
a_{i1}	a_{i2}	...	a_{ij}	...	a_{il}
...	...	...	...	...	...
a_{n1}	a_{n2}	...	a_{nj}	...	a_{nl}

Consider s different treatments of the same row i ($2 \le s \le l$) and associate with them s other treatments of a row i' ($i \ne i'$), respecting the correspondence between the treatments a_{ij} and $a_{i'j}$. Bringing together the 2s treatments in the same block and making all possible combinations, we obtain a partially balanced incomplete block design of size k = 2s.

Theorem 1:

The combinatory method (s) for $2 \le s < l$ associated with a rectangular association scheme, allow obtaining a rectangular PBIB design in three associated classes of parameters:

$$v = ln, b = n(n-1)C_l^s / 2, r = (n-1)C_{l-1}^{s-1}, k = 2s,$$
$$\lambda_1 = (n-1)C_{l-2}^{s-2}, \lambda_2 = C_{l-1}^{s-1}, \lambda_3 = C_{l-2}^{s-2}$$

Proof:

- The values of v and k are obvious
- r: For each treatment a_{ij}, we apply the procedure with the (l-1) other elements of the same row, there is C_{l-1}^{s-1} possibilities, each one repeated (n-1) times, so the treatment a_{ij} is repeated (n-1) C_{l-1}^{s-1} time in the b blocks of the design, this would be the value of r
- A couple of treatments a_{ij} and $a_{ij'}$ (j = j') of the same row i occurs together (n-1) C_{l-2}^{s-2} with the other couples. So = λ_1 (n-1) C_{l-2}^{s-2}
- A couple of treatments a_{ij} and $a_{i'j}$ (i' = i) of the same column j occurs together with C_{l-1}^{s-1} other couples. So $\lambda_2 = C_{l-1}^{s-1}$
- A couple of treatments a_{ij} and $a_{i'j'}$ (j' = j and i' = i) of rows and columns different occurs together C_{l-2}^{s-2} times. So $\lambda_2 = C_{l-2}^{s-2}$
- Finally b= n(n-1) $C_l^s / 2$

In fact: For each row, there is C_l^s possibilities and we apply the procedure with the (n -1) other rows. So, for the array we have n (n - 1) $C_l^s / 2$ blocks (we divide on 2 to avoid the repetition).

Remark 1:

The relations of existence of PBIB design (1) and (2) are checked:

$$rv = (n-1)C_{l-1}^{s-1} ln = n(n-1)C_l^s.s = bk$$

and:

$$\sum_{i=1}^{m} \lambda_i n_i = (n-1)C_{l-1}^{s-1}(s-1) + (n-1)C_{l-1}^{s-1} + C_{l-1}^{s-1}(s-1)$$
$$= (n-1)C_{l-1}^{s-1}(2s-1) = r(k-1)$$

Remark 2:

The more the s increases (s < l) the more the blocks' value decreases. This would provide us with economic designs comparing with the block numbers.

Example 1:

Example of rectangular design.
Let v = 12 = 3.4, (n = 3 and l = 4) treatments arranged in the following array:

1	2	3	4
5	6	7	8
9	10	11	12

Applying the combinatory method (s), s = 2 we obtain the following rectangular design with parameters:

$$v = 12, b = 18, r = 6, k = 4, \lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 1$$

1	2	5	6
1	2	9	10
1	3	5	7
1	3	9	11
1	4	5	8
1	4	9	12
2	3	6	7
2	3	10	11
2	4	6	8
2	4	10	12
3	4	7	8
3	4	11	12
5	6	9	10
5	7	9	11
5	8	9	12
6	7	10	11
6	8	10	12
7	8	11	12

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2.1. Algorithm of Construction of Rectangular Design

For n and l are large enough and for practical use, we propose an algorithm describing the method of construction of designs, by the combinatory method (s):

$$
\begin{cases}
\begin{cases}
\begin{cases}
\begin{cases}
\begin{cases}
\begin{cases}
\text{for } x_1 := 1 \ldots n-1 \text{ do} \\
\text{for } x_2 := x_1 + 1 \ldots n \text{ do} \\
\text{for } y_1 := 1 \ldots l-s+1 \text{ do} \\
\text{for } y_2 := y_1 + 1 \ldots l-s+2 \text{ do} \\
\text{for} \ldots \\
\text{for } y_s := y_{s-1} + 1 \ldots l \text{ do} \\
a_{x_1 y_1} \ a_{x_1 y_2} \ldots a_{x_1 y_s} \ a_{x_2 y_1} \ a_{x_2 y_2} \ldots a_{x_2 y_s} \\
\text{end} \\
\text{end} \\
\text{end} \\
\text{end} \\
\text{end} \\
\text{end}
\end{cases}
\end{cases}
\end{cases}
\end{cases}
\end{cases}
\end{cases}
$$

The combinatory method (s) can also be used for the construction of singular group divisible designs.

Lemma 1:

The combinatory method (s) for l = s associated with a group divisible association scheme, provide a singular Group Divisible (GD) design in two associated classes of parameters:

$$v = nl, b = n(n-1)/2, r = \lambda_1 = n-1, k = 2l, \lambda_2 = 1$$

Proof:

The design parameters are easily deduced from Theorem 1.

Example 2:

Example of singular group divisible design. Let v = 9 = 3.3, (n = 3 and l = 3) treatments arranged in the following array:

1	2	3
4	5	6
7	8	9

Applying the combinatory method (s), s = 2 we obtain the following rectangular design with parameters:

$$v = 9, b = 2, r = \lambda_1 = 2, k = 6, \lambda_2 = 1$$

1	2	3
4	5	6
7	8	9

2.2. Algorithm of Construction of GD Design

$$
\begin{cases}
\begin{cases}
\text{for } x_1 := 1 \ldots n-1 \text{ do} \\
\text{for } x_2 := x_1 + 1 \ldots n \text{ do} \\
a_{x_1 y_1} \ a_{x_1 y_2} \ldots a_{x_1 y_1} \ a_{x_2 y_1} \ a_{x_2 y_2} \ldots a_{x_2 y_1} \\
\text{end} \\
\text{end}
\end{cases}
\end{cases}
$$

3. DISCUSSION

The combinatory method (S) (2 < s ≤ l) which we propose, allows the construction of rectangular PBIB designs from only rectangular association schemes for v = nl treatments, thus the parameters of those obtained designs depend only on parameters of those association schemes. By against, the usual construction methods of rectangular designs require one hand, the existence of balanced incomplete blocks BIB designs (those designs do not necessarily exist for any parameters), on the other hand, the parameters of the rectangular PBIB designs obtained by the latter methods necessarily depend on the parameters of the initial BIB designs.

In addition, our method gives group divisible designs when the parameter s = l.

4. CONCLUSION

Note that our method is original, little exigent as it need only rectangular association schemes. It is also programmable and thus it is accessible to all users of experimental designs. Our combinatory method (S) can be considered as a basic tool to construct new PBIB designs associated with new rectangular right angular association schemes (m), m = 4, 5 and 7 in which these numbers of associated classes are rarely or not studied in the literature of experimental designs.

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