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Two Reformulations for the Dynamic Quadratic Assignment Problem

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Abstract: Problem statement: The Dynamic Quadratic Assignment Problem (DQAP), an NP-hard problem, is outlined and reformulated in two alternative models: Linearized model and logic-based model. **Approach:** The solution methods for both models based on combinatorial methods (Benders' Decomposition and Approximate Dynamic Programming) and constraint logic programming, respectively, are proposed. **Results:** Proofs of model equivalence and solution methodology are presented. **Conclusion:** Both proposed models are more simplified leading to possible hybrid adaptations of existing techniques for more practical approaches.

Key words: DQAP, linearized model, logic-based model, Benders, decomposition, dynamic programming, constraint logic programming

INTRODUCTION

A Dynamic Quadratic Assignment Problem (DQAP), mathematically formulated as a modified QAP, is defined as follows: given $n^4\times t$ cost coefficients c_{ijklt} (i, j, k, l = 1, 2,…, n and t = 1, 2,.., T), determine an $n^2\times t$ solution matrix $X = \| x_{ijt} \|$ so as to:

Minimize:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{t=1}^{T} C_{ijklt}X_{ijt}X_{klt} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{t=1}^{T-1} R_{ijlt}X_{ijt}X_{il(t+1)} \quad (1)$$

Subject to:

$$\sum_{i=1}^{n} X_{ijt} = 1 \text{ ,ij, t} \quad (2)$$

$$\sum_{j=1}^{n} X_{ijt} = 1 \text{ ,ii, t} \quad (3)$$

$$X_{ijt} \in \{0,1\}, \text{ii, j, t} \quad (4)$$

Where:

N = Represents the number of facilities/locations in each period t

T = Represents the number of discrete time periods

$C_{ijklt} = f_{ikt} * d_{jlt}$ = Represents the cost of assigning facility i to location j and facility k to location l at period t.

f_{ist} = The workflow cost from facility i to facility k at period t

d_{jlt} = The distance from location j to location l at period t

R_{ijlt} = Represents the rearranging cost when facility i located on location j at period t is moved to location l at period (t+1)

X_{ijt} = 1, if facility i is assigned to location j at period t. Otherwise, X_{ijt} is 0

Since the DQAP is NP-hard problem that is difficult to deal with in case of solving directly for an optimal solution. The objective of this study is to reformulate the DQAP into two alternative forms: Linearized model and logic-based model and to propose solution methods for both models.

MATERIALS AND METHODS

Linearized model: The DQAP can be linearized by defining $n^4\times t$ variables Y_{ijklt} and $n^3\times(t-1)$ variables $M_{ijl(t+1)}$:

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$$Y_{ijklt} \geq X_{ijt} + X_{klt} - 1$$

$$M_{ijl(t+1)} \geq X_{ijt} + X_{il(t+1)} - 1$$

Model 1: The linearized DQAP therefore becomes:

Minimize:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{t=1}^{T} C_{ijklt}Y_{ijklt} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{t=1}^{T-1} R_{ijlt}M_{ijl(t+1)} \quad (5)$$

Subject to:

$$Y_{ijklt} \geq X_{ijt} + X_{klt} - 1, i = 1,..,n,\ j = 1,\ldots,n, t = 1,\ldots T \quad (6)$$

$$M_{ijl(t+1)} \geq X_{ijt} + X_{il(t+1)} - 1, i = 1,..,n,\ j = 1,\ldots,n, t = 1,\ldots T \quad (7)$$

$$\sum_{i=1}^{n} X_{ijt} = 1, j = 1, \ldots, n, t = 1,\ldots T \quad (8)$$

$$\sum_{j=1}^{n} X_{ijt} = 1, i = 1,\ldots, n, t = 1,\ldots T \quad (9)$$

$$Y_{ijklt}, M_{ijl(t+1)} \geq 0, \forall\ i,\ j,\ t \quad (10)$$

$$X_{ijt} \in \{0,1\}, \forall\ i,\ j,\ t \quad (11)$$

RESULTS AND DISCUSSION

Extending the theorem and proof in Lawler (1963), it is possible to demonstrate that the linearization of DQAP is equivalent to DQAP. Let the DQAP defined in (1-4) be designated problem Q and the MILP defined in (5-11) be designated problem L. The following theorem assures the equivalence of Q and L for any given set of cost coefficients.

Theorem 1: The feasible solutions of problems Q and L can be placed in one-to-one correspondence with equal values of the cost functions. A feasible solution $X^{(Q)}$ of Q corresponds to a feasible solution ($X^{(L)}$,Y,V) of L if and only if $X^{(Q)}=X^{(L)}$.

Proof: It is sufficient to show that the constraints of problem L are such that for any given permutation matrix $X^{(L)}$ at a given period t, Y at period t and V from period t to t+1 are determined uniquely by the relations:

$$Y_{ijklt} = X_{ijt}X_{klt}$$

and:

$$M_{ijl(t+1)} = X_{ijt}X_{il(t+1)}$$

Since all of the variables are restricted to the value of 0 and 1, these relations are equivalent to:

$$Y_{ijklt} = 1 \Leftrightarrow X_{ijt} = X_{klt} = 1$$

and:

$$M_{ijl(t+1)} = 1 \Leftrightarrow X_{ijt} = X_{il(t+1)} = 1$$

It follows immediately from:

$$Y_{ijklt} \geq X_{ijt} + X_{klt} - 1$$

That:

$$Y_{ijklt} = 1 \Rightarrow X_{ijt} = X_{klt} = 1$$

and:

$$M_{ijl(t+1)} \geq X_{ijt} + X_{il(t+1)} - 1$$

That:

$$M_{ijl(t+1)} = 1 \Rightarrow X_{ijt} = X_{il(t+1)} = 1$$

In order to prove the converse, let $X_{ijt} = X_{klt} = 1$. Then, from the constraints:

$$Y_{ijklt} \geq X_{ijt} + X_{klt} - 1$$

$$M_{ijl(t+1)} \geq X_{ijt} + X_{il(t+1)} - 1$$

It follows that:

$$Y_{ijklt} \geq 1 \quad (12)$$

$$M_{ijl(t+1)} \geq 1 \quad (13)$$

Since the objective function is to minimize:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{t=1}^{T} C_{ijklt}Y_{ijklt} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{t=1}^{T-1} R_{ijlt}M_{ijl(t+1)}$$

and $C_{ijklt} \geq 0$ and $R_{ijlt} \geq 0$ by definition, Y_{ijklt} and $M_{ijl(t+1)}$ must choose the minimum feasible values accordingly to (12) and (13). Therefore, it follows that:

$$Y_{ijklt} = 1 \text{ and } M_{ijl(t+1)} = 1$$

Whenever:

$$X_{ijt} = X_{klt} = 1$$

The proposed solution method: For solving large scale MILP, Benders' Decomposition (BD) technique, which is presented by Benders (1962), can be applied. The algorithm solves a MILP problem via structure exploitation by decomposing a MILP into two problems-an integer master problem and a linear programming sub-problem-which are solved iteratively. Note that the solution of MILP in the master problem in this approach can further be approximated by the round-up of the solution from the relaxed linear assignment problem using the original Hungarian method. Dynamic Programming (DP) technique based on Rosenblatt (1986) can also be applied to the method in order to determine the sub-optimal solution and accelerate the convergence rate. BD generates a database of a subset of feasible solutions for DP to determine an approximate optimal solution. In order to accelerate BD, a trust-region constraint can be implemented into the master problem with a successive adaptation procedure (Muenvanichakul and Charnsethikul, 2009) to improve its performance.

Implementation of BD to linearized DQAP (5-11) lead to:

A linear programming sub-problem (dual problem):

Maximize:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{t=1}^{T}(X^*_{ijt} + X^*_{klt} - 1)U_{ijklt} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{t=1}^{T-1}(X^*_{ijt} + X^*_{il(t+1)} - 1)V_{ijl(t+1)} \tag{14}$$

Subject to:

$$0 \le U_{ijklt} \le C_{ijklt}, i = 1,..,n,\ j = 1,\dots,n, t = 1,\dots T \tag{15}$$

$$0 \le V_{ijl(t+1)} \le R_{ijlt}, i = 1,..,n,\ j = 1,\dots,n, t = 1,\dots T \tag{16}$$

for a given layout $X^*_{ijt} \in \{0,1\}, \forall\ i,\ j, t$ and

A mixed-integer-linear-programming master-problem:

$$\text{Minimize} \quad Z \tag{17}$$

Subject to:

$$Z \ge \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{t=1}^{T}(X_{ijt} + X_{klt} - 1)U^*_{ijklt} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{t=1}^{T-1}(X_{ijt} + X_{il(t+1)})V^*_{ijl(t+1)} \tag{18}$$

$$\sum_{i=1}^{n} X_{ijt} = 1, j\ =\ 1,\ \dots,\ n\ ,\ t\ =\ 1,\dots T \tag{19}$$

$$\sum_{j=1}^{n} X_{ijt} = 1, i\ =\ 1,\dots,\ n\ ,\ t\ =\ 1,\dots T \tag{20}$$

$$X_{ijt} \in \{0,1\}, \text{i i, j, t} \tag{21}$$

for given U^*_{ijklt} and $V_{ijl(t+1)}$.

The solution procedure starts by solving the sub-problem (14-16) from an initial layout X^*_{ijt} either from an initial guess value in the first iteration or a solution from the previous step (from the master problem) and then solving the master problem (17-21) from the solution of the sub-problem U^*_{ijklt} and $V_{ijl(t+1)}$. The procedure repeats until the different between the upper bound UB, the minimum of the current upper bound and the sub-problem objective value and the lower bound LB, the maximum of the current lower bound and the master problem objective value , is less than a given tolerance δ i.e.:

$$\text{UB-LB} \le \delta_{BD}$$

The cost of the total layout is the sub-problem objective value and the layout is X_{ijt} from the master problem.

Note that the constraint (18) in the master problem is corresponding to the optimality cut of the linearized DQAP. Theorem 2 assures that there exists no feasibility cut in BD of linearized DQAP (5-11); thereby the sub-problem (14-16) can be determined from (22) and (23) given below.

Theorem 2: There exists no feasibility cut in BD of the linearization DQAP.

Proof: It is sufficient to show that given X_{ijt}, X_{klt} for the corresponding the sub problem (14-16), to maximize the objective function (14):

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{t=1}^{T}(X^{*}_{ijt}+X^{*}_{klt}-1)U_{ijklt}$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{t=1}^{T-1}(X^{*}_{ijt}+X^{*}_{il(t+1)}-1)V_{ijl(t+1)}$$

It follows that $(X_{ijt}+X_{klt}-1)$ and $(X_{ijt}+X_{il(t+1)}-1)$ must be equal to either-1, 0 or 1 with $C_{ijklt}\geq 0$ and $R_{ijlt}\geq 0$ by definition. The optimal solution of the sub problem is forced to the maximum value of the bound as:

$$U^{*}_{ijklt}=C_{ijklt}\text{ if }(X_{ijt}+X_{klt}-1)=1 \qquad =0\text{ otherwise} \tag{22}$$

and:

$$V^{*}_{ijl(t+1)}=R_{ijlt}\text{ if }(X_{ijt}+X_{il(t+1)}-1)=1 \qquad =0\text{ otherwise} \tag{23}$$

Consequently, the problem is always feasible with no feasibility cut.

Logic-based model:
Model 2: A logic-based formulation of the DQAP with n×t variables $Y_{it}\times\{1, 2,.., n\}$. The constraint can be written as a set of in equations:

$$Y_{it}\neq Y_{kt}\ ,\forall i,k$$

with $i\neq k$ at period t.

It requires that $Y_{it},...,Y_{nt}$ be a permutation of 1,2,3,…,n at period t. These constraints can be only a single global constraint for each period as:

$$\text{all}-\text{different}\{Y_{1t},Y_{2t},Y_{3t},...,Y_{nt}\}\text{atperiod t}$$

where, n is a number of facilities and locations of the problem and t is a number of considering time periods.

Therefore, the logic-based model of DQAP is much more compact as:

Minimize:

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{t=1}^{T}F_{ikt}D_{y_{it}y_{kt}}+\sum_{i=1}^{n}\sum_{t=1}^{T-1}R_{Y_{it},Y_{i(t+1)}} \tag{24}$$

Subject to:

$$\begin{aligned}&\text{all}-\text{different}\{Y_{11},Y_{21},Y_{31},...,Y_{n1}\}\\&\text{all}-\text{different}\ \{Y_{12},Y_{22},Y_{32},...,Y_{n2}\}\\&\vdots\\&\text{all}-\text{different}\ \{Y_{1T},Y_{2T},Y_{3T},...,Y_{nT}\}\end{aligned} \tag{25}$$

Where:

$$C_{ijklt}=F_{ikt}\times D_{Y_{it},Y_{kt}}$$

In this form, the constraints with (n×T) real variables consist of entirely checkable constraints and search variables. A logic-based method can treat the all-different constraint directly without converting it to inequalities as well as constraint satisfaction seeks a feasible solution to a set of constraints.

It is possible to demonstrate that the logic-based model of DQAP is equivalent to DQAP. Let the DQAP defined in (5-11) be designated problem Q and the logic-based model of DQAP defined in (24-25) be designated problem L. The following theorem assures the equivalence of Q and L for any given set of cost coefficients.

Theorem 3: The feasible solutions of problems Q and L can be placed in one-to-one correspondence with equal values of the cost functions. A feasible solution $Y^{(Q)}$ of Q corresponds to a feasible solution ($Y^{(L)}$,all-different(Y)) of L if and only if $Y^{(Q)}=Y^{(L)}$.

Proof: It is sufficient to show that the constraints of problem L are such that for any given permutation matrix $Y^{(L)}$ at a given period t, Y at period t are determined uniquely by the relations:

$$\text{all}-\text{different}\{Y_{1t},Y_{2t},Y_{3t},...,Y_{nt}\}\text{atperiod t}$$

where, $Y_{it}\in\{1, 2,.., n\}$ to represent the facility i assigned to location and period Y_{it}

Since the constraint for each period t requires that $Y_{1t},Y_{2t},Y_{3t},...,Y_{nt}$ all take distinct values. It covers the idea that each facility is assigned exactly once to the location and vice versa. These relations are equivalent to:

$$Y_{it}\neq Y_{kt},\forall i,k\text{ with i¹k at period t}$$

It follows immediately as:

$$Y_{it}\neq Y_{kt}\Leftrightarrow Y_{i(y_{it})k(y_{kt})t}$$

$$Y_{i(y_{it})k(y_{kt})t}\geq X_{iy_{it}t}+X_{ky_{kt}t}-1$$

That:

$$Y_{i(y_{it})k(y_{kt})t} = Y_{ijklt} = 1 \Rightarrow X_{ijt} = X_{klt} = 1$$

$$M_{i(y_{it})(y_{i(t+1)})(t+1)} \geq X_{i(y_{it})} + X_{i(y_{i(t+1)})} - 1$$

That:

$$M_{i(y_{it})(y_{i(t+1)})(t+1)} = M_{ijl(t+1)} = 1 \Rightarrow X_{ijt} = X_{il(t+1)} = 1$$

In order to prove the converse, let $X_{ijt} = X_{klt} = 1$. Then, from the constraint of ILP:

$$Y_{ijklt} \geq X_{ijt} + X_{klt} - 1$$

$$M_{ijl(t+1)} \geq X_{ijt} + X_{il(t+1)} - 1$$

As proof in Theorem 1: then:

$$Y_{ijklt} = 1$$

Whenever:

$$X_{ijt} = X_{klt} = 1$$

Since:

$$X_{ijt} = X_{klt} = 1 \Rightarrow Y_{ijklt} = 1 = Y_{i(y_{it})k(y_{kt})t}$$

From the assignment constraints, therefore:

$$Y_{i(y_{it})} \neq Y_{k(y_{kt})}$$

And the logic-based constraint can take over to:

$$\text{all} - \text{different}\{Y_{1t}, Y_{2t}, Y_{3t}, ..., Y_{nt}\} \text{at period t}$$

The proposed solution method: For solving the logic based model, the structural algorithm of finding solution is therefore to branch on the search variables. It is impractical to keep branching until all search variables are determined. Logical inference as domain reduction algorithms can be applied to the checkable constraint before the variable domains becomes singletons. Then, Constraint Logic Programming (CLP) (Hooker, 2000) is a way of implementing the constraint satisfaction since integer programming methods cannot deal directly with an all-different constraint, $\{Y_{11}, Y_{21}, \ldots, Y_{n1}, \ldots, Y_{nt}\}$. It uses a programming language to specify at least the outline how the problem is to be solved. In this problem, ECLiPSe is a possible approach to solve the corresponding logic-based model. A tree search method in ECLiPSe is directly the way to find the solution without adding special inference. Especially, constraint propagation enforcing arc-consistency for general array expression, developed by Brand (2001), can be applied as the array constraint library to the program.

Detail implementation of the method and result are presented in Muenvanichakul (2009).

CONCLUSION

This study reformulated DQAP in two alternative forms: Linearized DQAP Model and Logic-Based DQAP Model. Both approaches leading to more simplified models can help developing more possible solution methods.

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