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K-Step Rational Runge-Kutta Method for Solution of Stiff System of Ordinary Differential Equations

P.O. Babatola, R.A. Ademiluyi and E.A. Areo
Department of Mathematical Sciences,
Federal University of Technology, Akure, Ondo State, Nigeria

Abstract: This study described the development, analysis and implementation of K-step implicit rational Runge-Kutta schemes for solution of stiff system of ordinary differential equations. Its development adopted taylor and binomial series expansion techniques to generate its parameters. The analysis of its basic properties adopted dalhquist a-stability model test equation and the results showed that the scheme was a-stable, consistent and convergent. Numerical results showed that the method was accurate and effective.

Key words: K-step, rational runge-kutta, a-stability, stiff ODEs

INTRODUCTION

A differential equation of the form:

$$y' = f(x, y),\ \ y(x_o) = y_o \tag{1}$$

whose Jacobian possesses eigen values:

$$\lambda_j = U_j + iV_j,\ j = 1(1)_n \tag{2}$$

where, $\sqrt{-1}$, satisfying the following conditions:

- $U_i << 0,\ j = 1(1)_n$
- $\mathrm{Max}\left|U_j(x)\right| >> \min\left|U_j(x)\right|$

$$\text{or } r(x) = \frac{\mathrm{Max}\left|U_j(x)\right|}{\min\left|U_j(x)\right|} >> 1 \tag{3}$$

$$y' = \lambda(y - E(x) + E^{1}(x),\ y(x_o) = y_o \tag{4}$$

where, E(x) is continuously differentiable, λ is a complex constant with Re(λ) <<0, with the exact solution:

$$y(x) = E(x) + y_o e^{\lambda x} \tag{5}$$

consisting of two components E(x) which is slowly varying in the interval of integration (x_o, b), and the second $y_o e^{\lambda x}$ component decaying rapidly in the transient phase at the rate of $-1/\lambda$ is midly stiff.

The system of differential equation:

$$y' = \begin{bmatrix} -0.00005 & 100 \\ -100 & -0.00005 \end{bmatrix} y \tag{6}$$

with $y(o) = [1,\ 1]^T, 0 \le \times \le 10\pi$ whose solution is obtained as:

$$y(x) = e^{-0.00005\,x \begin{bmatrix} \mathrm{Sin}100x & + & \mathrm{Cos}100x \\ \mathrm{Cos}100x & - & \mathrm{Sin}100x \end{bmatrix}} \tag{7}$$

Whose transitory phase is the entire interval of integration $0 \le \times \le 10\pi$ with 50π complete oscillation per unit interval is an ODEs possessing these types of properties are called stiff oscillating ODEs.

Most of the conventional Runge-Kutta schemes cannot effectively solve them because they have small region of absolute stability.

This perhaps motivated[8] to introduce a rationalized Runge-Kutta scheme of the form confirm existing phases in samples according to Emmanualson[4].

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{R} W_i K_i}{1 + y_n \sum_{i=1}^{R} V_i H_i} \tag{8}$$

Where:

Corresponding Author: P.O. Babatola, Department of Mathematical Sciences, Federal University of Technology, Akure, Ondo State, Nigeria

$$K_i = hf\left(x_n + c_i h, y_n + \sum_{i=1}^{S} a_{ij} k_j\right)$$

$$H_i = hg\left(x_n + d_i h, z_n + \sum_{i=1}^{S} b_{ij} k_j\right) \quad (9)$$

With

$$g(x_n, z_n) = -Z_n^2 f(x_{n,}\ y_n)$$

Subject to the constraints:

$$c_1 = \sum_{j=1}^{R} a_y$$

$$d_1 = \sum_{j=1}^{R} b_y \quad (10)$$

Since method possesses adequate stability property for solution of stiff ODEs, the papers consider the extension of the scheme to a general step process so that is can serve as a general purpose predictor for multistep schemes:

$$y_{n+m} = \frac{y_{n+m-1} + \sum_{i=1}^{R} W_i K_i}{1 + y_{n+m-1} \sum_{i=1}^{R} V_1 H_1} \quad (11)$$

Where:

$$K_1 = hf\left(x_{n+m-1} + C_1 h, y_{n+m-1} + \sum_{j=1}^{R} b_{ij} K_1\right)$$

$$H_1 = hg\left(x_{n+m-1} + d_1 h, Z_{n+m-1} + \sum_{j=1}^{R} b_{ij} H_1\right) \quad (12)$$

with

$$g(x_{n+kml}, Z_{n+k-1}) f(x_{n+m-1}, y_{n+m-1}) \quad (13)$$

In the spirit of Ademiluyi and Babatola[1] the scheme is classified into:

- Explicit if the constraints (10) is such that $a_{ij} = 0$ for $j \geq i$
- Semi-implicit if $a_{ij} = 0$ for $j > i$
- Implicit if $a_{ij} \neq 0$ for at least one $j > i$

MATERIALS AND METHODS

Derivation of the Method: In this research, the parameters Vi, Wi, Ci, di, a_{ij}, b_{ij} are to be determined from the system of non-linear equations generated by adopting the following steps:

- Obtained the Taylor series expansion of Ki's and Hi's about point (x_n, y_n) for i = 1(1)R
- Insert the series expansion into (10)
- Compare the final expansion with the Taylor series expansion of y_{n+1} about (x_n, y_n) in the power series of h

The number of parameters normally exceeds the numbers of equations, but in the spirit of[9], Gill [7] and Blum[3], these parameters are chosen as to ensure that (the resultant computation method has:

- Adequate order of accuracy of the scheme is achieved
- Minimum bound of local truncation error
- Large maximize interval of absolute stability
- Minimum computer storage facilities

One-step one-stage schemes: By setting M = 1 and R = 1, in Eq. 11 the general one-step one-stage scheme is of the form:

$$y_{n+1} = \frac{y_n + W_1 K_1}{1 + y_n V_1 H_1} \quad (14)$$

Where:

$$K_1 = hf(x_n + c_1 h,\ y_n + a_{11} K_1)$$

$$H_1 = hg(x_n + d_1 h,\ z_n + b_{11} H_1) \quad (15)$$

$$g(x_n\ z_n) = -Z_n^2 f(x_n, y_n) \quad (16)$$

and

$$Z_n = 1/y_n \quad (17)$$

with the constraints:

$$c_1 = a_{11}$$

$$d_1 = b_{11} \tag{18}$$

The binomial expansion theorem of order one on the right hand side of (10) yields:

$$y_{n+1} = y_n + W_1k_1 - y_n^2V_1H_1 + \text{(higher order terms)} \tag{19}$$

While The Taylor series expansion of y_{n+1} about y_n gives:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y'' + \frac{h^3}{6}y_n^{(3)} + \frac{h^4}{24}y_n^{(4)} + 0h^5 \tag{20}$$

Adopting differential notations:

$$\begin{aligned}
y'_n &= f_n \\
y''_n &= f_x + f_nf_y = Df_n \\
y_n^{(3)} &= f_{xx} + 2f_nf_{xy} + f_n^2f_{yy} + f_y\left(f_x + f_nf_y\right) \\
&= D^2f_n + f_yDf_n \\
y_n^{(4)} &= f_{xxx} + 3f_nf_{xxy} + 3f_n^2f_{xyy} + f_n^3f_{yyy} \\
&\quad + f_y\left(f_{xx} + 2f_nf_{xy} + f_n^2f_{yy}\right) \\
&\quad + \left(f_x + f_nf_y\right)\left(3f_{xy}\right) + \left(3f_nf_y + fy^2\right) \\
&= D^3f_n + f_yD^2f_n + 3Df_nDf_y + fy^2Df_n
\end{aligned} \tag{21}$$

substitute (21) into (20), we have:

$$\begin{aligned}
y_{n+1} &= y_n + hf_n + \frac{h^2}{2!}Df_n + \frac{h^3}{3!}\left(D^2f_n + f_yDf_n\right) \\
&+ \frac{h^4}{4!}\left(D^3f_n + f_yD^2f_n + 3Df_nDf_y + f_y^2Df_n\right) + 0h^5
\end{aligned} \tag{22}$$

Similarly the Taylor series expansion of K_1 about (x_ny_n) is:

$$K_1 = h\begin{pmatrix} f_n + \left(c_1hf_x + a_{11}k_1f_y\right) + ½ \\ \left(c_1^2h^2f_{xx} + 2c_1ha_{11}k_1f_{xy}\right) + a_{11}k_1^2f_{yy} \end{pmatrix} + 0(h^4) \tag{23}$$

Collecting coefficients of equal powers of h, Eq. 23 can be rewritten in the form:

$$K_1 = hA_1 + h^2B_1 + h^3D_1 + 0h^4 \tag{24}$$

where,

$$\begin{aligned}
&A_1 = f_n,.\ B_1 = c_1\ (f_x+f_nf_y) = C_1\ Df_n \\
&D_1 = c_1B_1f_y + ½\ C_1^2(f_{xx} + 2f_n f_{xy} + f_n^2f_{yy}) = Df_nf_y + ½D^2f_n
\end{aligned} \tag{25}$$

In a similar manner, expansion of H_1 about $g_{xxz} = \frac{-2f_{xx}}{y_n} + f_{xxy},\ g_{zz} = -2f_n - y_n^2f_{yy}$ (x_n, z_n) yields:

$$H_1 = hN_1 + h^2{}_M1 + h^3R_1 + 0h^4 \tag{26}$$

Where:

$$\begin{aligned}
N_1 &= g(x_n, z_n) = g_n \\
M_1 &= d_1\ (g_x + g_ng_z) = d_1Dg_n \\
R_1 &= d_1M_1g_z + ½d_1^2\left(g_{xx} + 2g_ng_{xz} + g_n^2g_{zz}\right) \\
&= d_1^2\left(g_zDg_n + ½D^2g_n\right)
\end{aligned} \tag{27}$$

$$\left.\begin{aligned}
&g_n = \frac{-f_n}{y_n^2},\ g_x = \frac{-f_x}{y_n^2},\ g_{xx} = \frac{-f_{xx}}{y_n^2} \\
&g_z = \frac{-2f_n}{y_n} + f_y,\ g_{xz} = \frac{-2f_x}{y_n} + f_{xy} \\
&g_{xzz} = -2f_x - 2y_n^2f_{xyy,} \\
&g_{zzz} = 4y_n^2f_y + 6y_n^2f_{yy} + y_n^4f_{yyy}
\end{aligned}\right\} \tag{28}$$

Substitute (28) into (26), we obtained:

$$\begin{aligned}
&N_1 = \frac{-f_n}{y_n^2},\ M_1 = \frac{-d_1}{y_n^2}\left(Df_n + \frac{2f_n^2}{y_n}\right) \\
&R_1 = \frac{d_1^2}{y_n^2}\left[\left(\frac{-2f_n}{y_n} + f_y\right)\left(Df_n + \frac{f_n^2}{y_n}\right)\right] \\
&\quad + ½\left(D^2f_n - \frac{2f_n}{y_n}\left(\frac{f_n^{\ 2}}{y_n} + f_x\right)\right]
\end{aligned} \tag{29}$$

Using (25) and (26) in (19), to get:

$$\begin{aligned}
y_{n+1} &= y_n + W_1\left(hA_1 + h^2B_1 + h^3D_1 + 0h^4\right) \\
&\quad - y_n^2\left(V_1\left(hN_1 + h^2M_1 + h^3R_1 + 0h^4\right)\right] \\
&= y_n\left(W_1A_1 - y_n^2V_1N_1\right)h + \left(W_1B_1 - y_n^2V_1M_1\right)h^2 \\
&\quad + \left(W_1D_1 - y_n^2V_1R_1\right) + 0h^4
\end{aligned} \tag{30}$$

Comparing the coefficients of the powers of h in Eq. 22 and 30, we obtained:

$$W_1A_1 - y_n^2V_1N_1 = f_n \tag{31}$$

Since

$A_1 = f_n N_1 = \frac{-f_n}{y_n^2}$, Eq. 31 yields:

$$W_1 + V_1 = 1 \quad (32)$$

Similarly from coefficients of h^2 in Eq. 22 and 30, we have:

$$W_1B_1 - y_n^2 V_1M_1 = Df_n \quad (33)$$

Also from (22) and (31) we obtained:

$$(W_1C_1 + V_1d_1)\left(Df_n + \frac{2f_n^2}{y_n}V_1d_1\right) = \frac{Df_n}{2} \quad (34)$$

$$W_1C_1 + V_1d_1 = ½ \quad (35)$$

Putting Eq. 35 and 32 together, we have a system of non linear simultaneous equations:

$$\begin{aligned} W_1 + V_1 &= 1 \\ W_1C_1 + V_1d_1 &= ½ \end{aligned} \quad (36)$$

with the constraints

$$\begin{aligned} a_{11} &= c_1 \\ b_{11} &= d_1 \end{aligned} \quad (37)$$

Solving Eq. 37 and 38, we obtained:

$$V_1 = W_1 = ½\ c_1 = a_{11} = ¾,\ d_1 = b_{11} = ¼$$

Substituting these values in Eq. 14 we obtain a family of one-step, one stage schemes of the form:

$$y_{n+1} = \frac{y_n + \frac{1}{2}K_1}{1 + \frac{y_n}{2}H_1} \quad (38)$$

Where:

$$K_1 = hf(x_n + ¾\ h,\ y_n + ¾\ K_1)$$

$$H_1 = hg(x_n + ¼\ h,\ z_n + ¼\ H_1) \quad (39)$$

with

$V_1 = ¾$, $W_1 = ¼$, $d_1 = c_1 = ½$, $a_{11} = b_{11} = ½$ Eq. 20 becomes:

$$y_{n+1} = \frac{y_n + \frac{1}{4}K_1}{1 + \frac{3}{4}y_nH_1} \quad (40)$$

Where:

$$K_1 = hf(x_n + ½h,\ y_n + ½\ K_1)$$

$$H_1 = hg(x_n + ½\ h,\ z_n + ½\ H_1) \quad (41)$$

with

$$W_1 = 1/3,\ V_1 = 2/3,\ a_{11} = C_1 = 1/3,\ b_{11} = d_1 = 7/12$$

$$y_{n+1} = \frac{y_n + \frac{1}{3}K_1}{1 + \frac{2}{3}y_nH_1} \quad (42)$$

Where:

$$\begin{aligned} K_1 &= hf(x_n + 1/3\ h,\ y_n + 1/3\ K_1) \\ H_1 &= hg(x_n + 7/12\ h,\ x_n + 7/12\ H_1) \end{aligned} \quad (43)$$

The basic properties of the method: The basic properties required of a good computational method for stiff ODEs includes consistency, convergence and stability and A-stability.

Consistency: A scheme is said to be consistent, if the difference equation of the computation formula exactly approximate the differential equation it intends to solve[2].

To prove that Eq. 11 is consistent.

Recall that:

$$y_{n+k} = \frac{y_{n+k-1} + \sum_{i=1}^{R} W_iK_i}{1 + y_{n+k-1}\sum_{i=1}^{R} V_iH_i} \quad (44)$$

Subtract y_{n+k-1} on both sides of Eq. 44:

$$y_{n+k} - y_{n+k-1} = \frac{y_{n+k-1} + \sum_{i=1}^{R} W_iK_i}{1 + y_{n+k-1} + \sum_{i=1}^{R} V_iH_i} - y_{n+k-1} \quad (45)$$

$$\frac{\sum_{i=1}^{R} W_iKi - y_{n+k-1}^2 \sum V_iH_i}{1 + y_{n+k-1}\sum_{i=1}^{R} V_iH_i} \quad (46)$$

with

$$K_i = hf\left(x_{n+k-1} + C_i h,\ y_n + \sum_{j=1}^{R} a_{ij} k_j\right)$$

$$H_i = hg\left(x_{n+k-1} + d_i h,\ Z_n + \sum_{j=1}^{R} b_{ij} H_j\right) \quad (47)$$

Substituting (48) into (47), dividing through by h and taking limit as h tends to zero, obtain:

$$\frac{\lim_{h\to o} y_{n+k} - y_{n+k-1}}{h} = \sum_{i=1}^{R} W_i f(x_{n+k-1},\ y_{n+k-1}) - y^2_{n+k-1} \sum V_i g(x_{n+k-1}, y_{n+k-1}) \quad (48)$$

but

$$g(x_{n+k-1}, Z_{n+k-1}) = \frac{1}{y^2_{n+k-1}} f\left(x_{n+k-1,}\ y_{n+k-1}\right) \quad (49)$$

then

$$\text{Lt} \quad \frac{y_{n+k} - y_{n=k-1}}{h} = \sum_{i=1}^{R} (W_i + V_i) f\left(x_{n+k-1,}\ y_{n+k-1}\right), \quad \text{but}$$

$$\left(\sum_{L=1}^{R} W_i + V_i\right) = 1$$

$$y'(x_{n+k-1}) = f(x_{n+k-1}, y_{n+k-1}) \quad (50)$$

Hence the method is consistent.

Convergence: Since the proposed scheme is one –step, the numerical scheme (11) for solving ODEs (1) is said to be convergent, and it is consistent, by Lambert (1973) when it is applied to initial value problem (1) generated a corresponding approximation y_n which tend to the exact solution $y(x_n)$ as n approaches infinity, that is:

$$y_n \to y(x_n) \text{ as } n \to \infty$$

Let e_{n+k} and T_{n+k} denote the discretization and Truncation errors generated by (1) respectively. Adopting Binomial expansion and ignoring higher term, Eq. 1:

$$y_{n+k} = y_{n+k-1} + \sum_{i=1}^{R} W_i K_i - y^2_{n+k-1} \sum_{i=1}^{R} V_i H_i + \text{(higher order term)} \quad (51)$$

If $y(x_{n+k})$ is the approximate theoretical solution, it seen to satisfy the difference equation:

$$y(x_{n+k}) = y(x_{n+k-1}) + \sum_{i=1}^{R} W_i K_i - y^2_{(n+k-1)} \sum_{i=1}^{R} V_i H_i + \text{(higher order term)} + T_{n+k} \quad (52)$$

Subtract Eq. 51 from 52:

$$\begin{aligned} y_{n+k} - y(x_{n+k}) &= y_{n+k-1} - y(x_{n+k-1}) \\ &+ h\left[\psi_2\left(x_n y_n^{(x_n)} h\right) - \psi_2(x_n,\ y_n; h)\right] \\ &+ h\left(\phi_1(x_n, y(x_n); h) - \phi_1\left(x_n, y_n; h\right)\right) + T_{n+k} \end{aligned} \quad (53)$$

where, $\psi_2(x_n(y(x_n);h)$ are assumed to be continuous functions in the domain:

$a \le x \le b$, $|y| < \infty$, $0 \le h \le h_o$ defined as:

$$\begin{aligned} h\phi_1\left(x_{n+k-1} y(x_n); h\right) &= \sum_{i=1}^{R} W_i K_i h \phi_2\left(x_{n+k_y;}\ Z(x_{n+k-1})h)\right) \\ = \sum_{i=1}^{R} V_i H_i &= \frac{-h}{y^2_{(x_n+k-1)^{\psi_2}}} \left(x_{n+k-1} y_{n+k-1}\right) \end{aligned} \quad (54)$$

This Eq. 54 modifies into:

$$\begin{aligned} e_{n+k=} e_{k+k-1} &+ h\left[\psi_2\left(x_{n+k-1},\ y(x_{n+k-1})\right); h\right] \\ &- \psi_2(x_{n+k-1},\ y_{n+k-1}; h) + h\left\{\left[\phi_1 y(x_{n+k=1}\right); h\right) \\ &- \phi_1(x_{n+k-1}, y_{n+k-1}; h) + T_{n+k} \end{aligned} \quad (55)$$

By taking the absolute value on both sides of Eq. 61 we have inequality:

$$|e_{n+k}| \le e_{n+k-1} + KH|e_{n+k-1}| + hL|e_{n+k-1}| + T \quad (56)$$

where, L and K are the Lipschitz constant for $\phi_1(x,y;h)$ and ψ_2 (x,y;h) respectively and:

$$T = |T_{n+k}|_{a \le x \le b} \quad (57)$$

By setting N = L+ K

Inequality (57) becomes:

$$|e_{n+k}||e_{n+k-1}|(1 + hN) + T \quad (58)$$

By adopting this theorem on convergence of sequence of real numbers quoted without proof from[7], that is:

If $\{e_j,\ j = o(1)_{n+k}\}$ be set of real number. If there exist finite constants R and S such that:

$$\left|e_i\right| \le R\left|e_{i-1}\right| + S,\ i = 1(1)_{n+k-1} \tag{59}$$

$$\text{then}\left|e_i\right| \le \left(\frac{R^i - 1}{R - 1}\right)s + R^i\left|e_o\right|, R \ne 1 \tag{60}$$

Thus (59) becomes:

$$\left|e_{n+k}\right| \le \frac{(1+hN)^{n+k}_{-1}}{hN} + (1+hN)^{n+k}\left|e_o\right| \tag{61}$$

since

$$\left|1+hN\right|^{n+k} = e^{(n+k)hN} = e^{N(x_{n+k}-a)} \tag{62}$$

and

$$x_{n+k} \le b, \text{ then } x_{n+k} - a \le b - a$$

Consequently:

$$e^{N(xn+k-a)} \le e^{N(b-a)} \tag{63}$$

$$\left|e_{n+k}\right| \le \left(\frac{e^{N(b-a)} - 1}{hN}\right)T + e^{N(b-a)}\left|e_o\right| \tag{64}$$

Considering Eq. 64 and adopting first mean value theorem:

$$\begin{aligned} T_{n+k} &= h\begin{bmatrix} \psi_2(x_{n+k-1} + \theta h,\ y(x_{n+k-1} + \theta h)) - \psi_2(x_{n+k-1} \\ + y(x_{n+k-1}) + h\left[\phi_1(x_{n+k-1} + 0h, y(x_{n+k-1} + 0h)\right] \end{bmatrix} \\ &\quad - \phi_1(x_{n+k-1}, y(x_{n+k-1})))\big] \\ &= h\begin{bmatrix} \psi_2(x_{n+k-1}\theta h,\ y(x_{n+k-1} + \theta h)) - \psi_2(x_{n+k-1} + \theta h, \\ y(x_{n+k-1}) + \psi_2(x_{n+k-1} + \theta h, (x_{n+k-1} + \theta h, y \\ (x_{n+k-1} + \theta h) \end{bmatrix} \\ &\quad - \phi_1(x_{n+k-1} + \theta h,\ y(x_{n+k-1}))\big], 0 \le \theta \le 1 \end{aligned} \tag{65}$$

By taking the absolute value of (55) on (66) both sides Eq. 46 into consideration, we have:

$$\begin{aligned} T &= hL\left|y((x_{n+k-1} + 0h) - y(x_n)_{k-1}\right| + jh^2\theta \\ &+ hk\left|y(x_{n+k-1} + \theta h - y(x_{n+k-1})\right| + Mh^2\theta \end{aligned} \tag{66}$$

where, M and j are the partial derivative of ϕ_1 and ψ_1 with respect of x respectively.

By setting Q = j + m and

$$y = \sup_{a \le x \le b}(y'(x)) \tag{67}$$

Therefore, Eq. 54 yields:

$$T = h^2\theta\ (NY + Q) \tag{68}$$

By substituting (57) into (52), we have:

$$\left|e_{n+k}\right| \le h^2\theta e^{N(b-a)}\ (NY + Q) + e^{N(b-a)}\left|e_o\right| \tag{69}$$

Assuming no error in the input data, that is $e_o = 0$.

Then the limit as h tends to zero, we obtain in Eq. 70 yields:

$$\lim_{\substack{h \to o \\ n \to \infty}} \left|e_{n+k}\right| = 0$$

which implies that:

$$\lim_{\substack{h \to o \\ n \to \infty}} y_n = y(x_n)$$

Thus establishing the convergence of scheme (11).

Stability Properties: To analyse the stability property of this schemes, apply scheme (1) to Dalhquist[4] stability scalar test initial value problem:

$$y' = \lambda y,\ y(x_o) = y_o \tag{70}$$

to obtained a difference equation:

$$y_{n+k} = \mu(z)y_{n+k-1} \tag{71}$$

with the stability function:

$$\mu(z) = \frac{1 + ZW^T(I - ZA)^{-1}e}{1 + ZV^T(I - ZB)^{-1}e} \tag{72}$$

Where:

$$W^T = (W_1, W_2 \ldots W_r)$$
$$V^T = (V_1, V_2 \ldots V_r)$$

To illustrate, this we consider the one-step, one-stage scheme:

$$y_{n+1} = \frac{y_n + W_1 k_1}{1 + y_n V_1 H_1} \tag{73}$$

Where:

$$K_1 = hf(x_n + c_1 h, y_n + a_{11}K_1)$$

$$H_1 = hg(x_n + d_1 h, z_{n+} b_{11}H_1) \qquad (74)$$

Applying (73) to the stability test Eq. 70 we obtain the recurrent relation:

$$y_{n+1} = \mu(z) y_n \qquad (75)$$

with stability function:

$$\overline{\mu}(z) = \frac{1 + W_i Z(1 - a_{11}Z)^{-1}}{1 - V_i Z(1 + b_{11}Z)^{-1}} \qquad (76)$$

For example, to analyze scheme (74) will introduce a convergent and stable approximation to the solution of stability function if:

$$|\mu(z)| = \left|\frac{1 + \frac{1}{4}Z}{1 - \frac{3}{4}Z}\right| \prec 1 \qquad (77)$$

That is (-∞ <x < o), the scheme is A-stable because the interval of absolute stability is (-∞, o).

RESULTS AND DISCUSSION

In order to access performance of the schemes the following sample problems were solved, with the schemes adopting Fehlberg[6] approach.

Problem 1: Consider the stiff system of ODEs of the form:

$$Y' = AY \qquad (78)$$

$$\text{Where } A = \begin{bmatrix} 1.0 & -4.99 & 0 \\ 0 & -5.0 & 0 \\ 0 & 2.0 & -12 \end{bmatrix} \qquad (79)$$

With initial condition y(o) = (2, 1, 2).

In the theoretical interval $0 \le x \le 1$. Its general solution is:

$$\begin{pmatrix} y_1(x) & = e^{-x} + e^{-5x} \\ y_2(x) & = e^{-5x} \\ y_3(x) & = e^{-5x} + e^{-12x} \end{pmatrix}$$

Its numerical solution is found in Table 1.

Problem 2: The second sample problem considered is the stiff system of initial values problems of ODEs below:

$$y = \begin{pmatrix} -0.5 & 0 & 0 & 0 & 0 \\ 0 & -1.0 & 0 & 0 & 0 \\ 0 & 0 & -9.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10.0 \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \text{with initial condition}$$

y (0) = [1 1 1 1]

The results are shown in Table 2.

Table 1: Numerical result of k-step implicit rational runge-kutta schemes for solving stiff systems of ordinary differential equations

X	Control step size	Y1 / E1	Y2 / E2	Y3 / E3
		0.1980099667D+01	0.9706425830D+00	0.8869204674D+00
0.3000000000D-01	0.3000000000D-01	0.8291942688D-09	0.3281419103D-07	0.8161313500D-05
		0.1885147337D+01	0.8379203859D+00	0.4917945068D+00
0.1774236000D+00	0.1771470000D-01	0.9577894033D-01	0.3422855333D-08	0.5357828618D-06
		0.1791235536D+01	0.7191953586D+00	0.2663621637D+00
0.3307246652D+00	0.1046033532D-01	0.11050933794D-10	0.35587255336D-09	0.3474808041D-07
		0.1694213422D+01	6088845946D+00	0.1365392880D+00
0.4977858155D+00	0.6176733963D-02	0.1269873096D-11	0.3655098446D-10	0.2146555961D-08
		0.1556933815D+01	0.4729421983D+00	0.4953161076D-01
0.7512863895D+00	0.3647299638D-01	0.1425978891D-08	0.3505060447D-07	0.1010194837D-05
		0.1435390902D+01	0.3709037123D+00	0.1867601194D-01
0.9951298893D+00	0.2153693963D-01	0.1594313570D-09	0.3316564301D-08	0.4481540687D-07

Table 2: Numerical result of k-step implicit rational runge-kutta schemes for solving stiff systems of ordinary differential equations

		Y1	Y2	Y3	Y4
X	Control step size	E1	E2	E3	E4
		0.9950124792D+00	0.9900498337D+00	0.9139311928+00	0.9048374306D+00
0.3000000000D-01	0.3000000000D-01	0.2597677629D-10	0.4145971344D-09	0.2617874150D-05	0.3971726602D-05
		0.9708623323D+00	0.9425736684D+00	0.5872698932D+00	0.5535451450D+00
0.1774236000D+00	0.1771470000D-01	0.3078315380D-11	0.4788947017D-10	0.2005591107D-06	0.2890213078D-06
		0.9402798026D+00	0.8841261072D+00	0.3300866691D+00	0.2918382654D+00
0.3694667141D+00	0.1046033532D-01	0.3621547506D-12	0.5454525720D-11	0.1355160001D-07	0.1829417523D-07
		0.9144602205D+00	0.8362374949D+00	0.1999708940D+00	0.1672231757D+00
0.5365278644D+00	0.6176733963D-02	0.4285460875D+13	0.6268319197D-12	0.9915873955D-09	0.1265158728D-08
		0.8693495443D+00	0.7557686301D+00	0.8044517344D-01	0.6079796167D-01
0.8400599835D+00	0.3647299638D-01	0.4961209221D-10	0.6922001861D-09	0.5087490103D-06	0.5899525189D-06

CONCLUSION

From the above results (Table 1 and 2), it can be seen that the proposed schemes are quite accurate, convergent and stable.

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