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On a Class of Nonhomogeneous Fields in Hilbert Space

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Abstract: Two-parametric semigroups of operators in Hilbert space with bounded infinitesimal doubly commuting operators are studied. The characteristics describing deviation of a semigroup from unitary one, when infinitesimal operators are unitary, in particular, nonunitary index, have been introduced. Necessary and sufficient conditions for nonunitary index finiteness have been obtained.

Keywords: Nonhomogeneous Fields, Multi-parametric Semigroup, Doubly Commuting Operators

INTRODUCTION

One-parametric semigroups of operators were studied adequately, both from theoretical and applied pointviews [1], A few works in harmonic analysis are devoted to study multi-parametric semigroups [2, 3]. We study the nonhomogeneous field $u(x_1,x_2)$ in Hilbert space *H* which is presented in the form

$$u(x_1,x_2) = e^{ix_1T_1+ix_2T_2}u_0,$$

where $u_0 \in H$, T_1 and T_2 are bounded doubly commuting operators [4]. Consider a scalar product

$$\langle u(x_1,x_2),u(y_1,y_2)\rangle_H = K(x_1,y_1;x_2,y_2).$$

Then if $T_j = T_j^*(j=1,2)$, the function $K(x_1,y_1;x_2,y_2)$ depends only on corresponding differences $K(x_1-y_1;x_2-y_2)$ and the field is homogenous.

If $T_1 \neq T_1^*$ or $T_2 \neq T_2^*$ or both operators $T_j\,(j=1,2)$ are non self-adjoint operators, then the field $u(x_1,x_2)$ is nonhomogeneous. In addition, if $T_j\,(j=1,2)$ belongs to a certain class of non self-adjoint operators, one may invoke spectral theory of doubly commuting non self-adjoint operators to study the field $u(x_1,x_2)$.

Functional Characteristic of the Nonhomogeneous Field: Consider the case when $T_j\,(j=1,2)$ are doubly commuting unitary or quasi-unitary operators and introduce some numerical and functional characteristics, describing deviation of the field in the form

$$u(x_1,x_2) = e^{ix_1T_1+ix_2T_2}u_0,$$

where T_j are unitary operators. Note that for unitary doubly commuting operators (we call the corresponding field to be unitary) function $K(x_1,y_1;x_2,y_2)$ may be presented in the form

$$K(x_1-y_1;x_2-y_2;x_1+y_1;x_2+y_2) = \int_0^{2\pi} e^{i(x_1-y_1)\cos f_1(\lambda)+i(x_2-y_2)\cos f_2(\lambda)} \times e^{-(x_1+y_1)\sin f_1(\lambda)-(x_2+y_2)\sin f_2(\lambda)}dF_\lambda, \quad (1)$$

where, $f_k(\lambda)$ real-value functions, $\Delta F_\lambda = \langle \Delta E_\lambda u_0, u_0\rangle$, and E_λ is the spectral function of unitary operator

$$T_0 = \int_0^{2\pi} e^{i\lambda}dE_\lambda.$$

The above form of *K* follows from the Neuman theorem for generating operator T_0 of a set of mutually commuting selfadjoint (unitary) operators [5].

Taking into the account the well-known fact for commuting operators T_1 and T_2 one of them is a function of another [5]. It is not difficult to verify that if T_1 and T_2 are the unitary commutative operators then the function $K(x_1,y_1;x_2,y_2)$ satisfies the following equation

$$L_{x_jy_j}K(x_1,y_1;x_2,y_2) = 0, \qquad (j=1,2) \quad (2)$$

where

$$L_{xy} = I - \frac{\partial^2}{\partial x \partial y}.$$

From the applied point of view $K(x_1, y_1; x_2, y_2)$ is the correlation function for some random field, because $K(x_1, y_1; x_2, y_2)$ is Hermitian nonnegative function. Hence there exists Gaussian normal field for which $K(x_1, y_1; x_2, y_2)$ is the correlation function and the results obtained may be interpreted as a correlation theory for nonhomogeneous random field. Here after we will consider that

$$H = H_u = \overline{\bigvee_{x_1, x_2 \geq 0} T^{x_1} T^{x_2} u_0}, \quad (x_j \text{ are integers}).$$

Let us consider the field

$$u^*(x_1, x_2) = e^{ix_1 T_1^* + ix_2 T_2^*} u_0,$$

which, henceforth, we will call it the adjoint field.

It is obvious that for the field $e^{-ix_1 T_1 + ix_2 T_2} u_0$ (T_1 and T_2 double commuting operators) to be unitary it is necessary and sufficient that K should be in accordance with

$$L_{x_j y_j} K(x_1, y_1; x_2, y_2) = 0, \qquad (j = 1, 2)$$

Lemma 1: *Let* $H_u = H_u^* = H$, *and* $u(x_1, x_2) = e^{ix_1 T_1 + ix_2 T_2} u_0$. *Then the necessary and sufficient for* T_1 *and* T_2 *to be commutative is that*

$$\frac{\partial^2}{\partial x_1 \partial y_2} \widetilde{K}(x_1, y_1; x_2, y_2) = \frac{\partial^2}{\partial x_2 \partial y_1} \widetilde{K}(x_1, y_1; x_2, y_2),$$

where

$$\widetilde{K}(x_1, y_1; x_2, y_2) = \langle u(x_1, x_2), u^*(y_1, y_2) \rangle.$$

The lemma proof follows from the definition of the function, $\widetilde{K}(x_1, y_1; x_2, y_2)$ and a relationship

$$\frac{\partial^2 \widetilde{K}}{\partial x_\ell \partial y_m} = -\langle T_\ell T_m u(x_1, x_2), u^*(y_1, y_2) \rangle.$$

If $L_{x_1 y_1} L_{x_2 y_2} K(x_1, y_1; x_2, y_2) \neq 0$, then the function

$$W(x_1, y_1; x_2, y_2) = L_{x_1 y_1} L_{x_2 y_2} K(x_1, y_1; x_2, y_2) \quad (3)$$

may be considered as a functional characteristic of deviation infinitesimal commutative operators T_1 and T_2 from unitary operators.

If T_1 and T_2 are doubly commuting operators ($([T_1, T_2] = 0,\ [T_1, T_2^*] = 0)$, then from (3) we may obtained the following presentations for W:

$$W(x_1, y_1; x_2, y_2) = \langle (I - T_1^* T_1)(I - T_2^* T_2) u(x_1, x_2), u(y_1, y_2) \rangle. \quad (4)$$

The presentation (4) is significant for further studies.

Remark 1: *To reconstruct* $K(x_1, y_1; x_2, y_2)$ *by* $W(x_1, y_1; x_2, y_2)$ *one may solve Darboux-Goursat problem for equation*

$$L_{x_1 y_1} L_{x_2 y_2} K(x_1, y_1; x_2, y_2) = W(x_1, y_1; x_2, y_2)$$

twice, and defining appropriate conditions additionally.

Remark 2: *If the operators* T_1 *and* T_2 *are commuting operators, but are not doubly commuting, then*

$$W(x_1, y_1; x_2, y_2) = \langle (I - T_1^* T_1 - T_2^* T_2 + T_2^* T_1^* T_1 T_2) u(x_1, x_2), u(y_1, y_2) \rangle$$

and further analysis is based on assumption of commutant $[T_1, T_2^*]$ *properties, for example* T_1, T_2^* *and* $[T_1, T_2^*]$ *form Lie algebra.*

Theorem 1: *If dim* $H_0 = r < \infty$, *where*

$$H_0 = \overline{(I - T_1^* T_1) H} \cap \overline{(I - T_2^* T_2) H},$$

then

$$W(x_1, y_1; x_2, y_2) = \sum_{\alpha=1}^{r} \lambda_\alpha \Phi_\alpha(x_1, x_2) \overline{\Phi_\alpha(y_1, y_2)}, \quad (5)$$

where $\Phi_\alpha(x_1, x_2) = \langle u(x_1, x_2), h_\alpha \rangle$, $h_\alpha \in H_0$, *and* λ_α *are real numbers.*

Proof: Consider the orthonormal basis $\{h_\alpha\}_{\alpha=1}^{r}$ in H_0, consisting of eigenvector contraction of self-adjoinet operator $(I - T_1^* T_1)(I - T_2^* T_2)$ onto its invariant subspace H_0. Since

$$B_H = (I - T_1^*T_1)(I - T_2^*T_2)u(x_1,x_2)$$
$$= \sum_{\alpha=1}^{r} \langle Bu(x_1,x_2), h_\alpha \rangle h_\alpha$$
$$= \sum_{\alpha=1}^{r} \langle u(x_1,x_2), Bh_\alpha \rangle h_\alpha$$
$$= \sum_{\alpha=1}^{r} \lambda_\alpha \langle u(x_1,x_2), h_\alpha \rangle h_\alpha,$$

where $Bh_\alpha = \lambda_\alpha h_\alpha$ and λ_α are eigenvalues of the operator B.

As a result, we obtain

$$W(x_1,y_1;x_2,y_2) = \sum_{\alpha=1}^{r} \lambda_\alpha \Phi_\alpha(x_1,x_2)\overline{\Phi_\alpha(y_1,y_2)} . \square$$

Remark, that the function $K(x_1,y_1;x_2,y_2)$ defines the Hilbert-valued function $u(x_1,x_2)$ quite completely. The next assertion is valid.

Lemma 2: *Consider the two functions $u_1(x_1,x_2)$ and $u_2(x_1,x_2)$ with values belonging to the Hilbert spaces $H_{uj} = \overline{\bigvee_{x_1,x_2 \ge 0} u_j(x_1,x_2)}$ respectively, where the scalar product is generated by the respective function*

$$K(x_1,y_1;x_2,y_2) = \langle u_j(x_1,x_2), u_j(y_1,y_2) \rangle_{H_j}$$
$$= K_j(x_1,y_1;x_2,y_2).$$

If $K_1(x_1,y_1;x_2,y_2) = K_2(x_1,y_1;x_2,y_2)$, then there exists a unitary transformation $U \in [H_1,H_2]$ such that $u_2(x_1,x_2) = Uu_1(x_1,x_2)$. Moreover if $u(x_1,x_2) = e^{ix_1T_1 + ix_2T_2}u_{0_1}$, then $u_2(x_1,x_2)$ is also generated by two-parametric semigroup of operators $u_2(x_1,x_2) = e^{ix_1B_1 + ix_2B_2}u_{0_2}$.

Proof: Consider lineals

$$L_j = \left\{ \sum_{\alpha,\beta=1}^{n_1,n_2} C_{\alpha,\beta} u_j(x_\alpha, x_\beta) \right\} n_1, n_2 < \infty,$$

where, $C_{\alpha,\beta}$ are complex numbers. For $h_1^{(j)}, h_2^{(j)} \in L_j$ define binary form

$$\langle h_1^{(j)}, h_2^{(j)} \rangle_{L_j} = \sum_{\alpha,\beta=1}^{n_1,n_2} \sum_{p,q=1}^{m_1,m_2} C_{\alpha,\beta} Q_{p,q} K_j(x_\alpha, y_p; x_\beta, y_q),$$

where,

$$h_1^{(j)} = \sum_{\alpha,\beta=1}^{n_1,n_2} C_{\alpha,\beta} u_j(x_\alpha, x_\beta),$$

$$h_2^{(j)} = \sum_{p,q=1}^{m_1,m_2} Q_{p,q} u_j(x_p, x_q).$$

Then L_j become pre-Hilbert spaces. Define isometric (by virtue of equality $K_1(x_1,y_1;x_2,y_2) = K_2(x_1,y_1;x_2,y_2)$), transformation of L_1 into L_2:

$$U\left(\sum_{\alpha,\beta=1}^{n_1,n_2} C_{\alpha,\beta} u_1(x_\alpha, x_\beta) \right) = \left(\sum_{\alpha,\beta=1}^{n_1,n_2} C_{\alpha,\beta} u_2(x_\alpha, x_\beta) \right).$$

Extending U for closures L_1 and L_2 we get the first assertion of the Lemma. The second part of the Lemma follows immediately from the evident relationships:

$$u_2(x_1,x_2) = Uu_1(x_1,x_2) = Ue^{ix_1T_1 + ix_2T_2}u_{0_1} = e^{ix_1B_1 + ix_2B_2}u_{0_2},$$

where $B_j = UT_jU^{-1}, u_{0_2} = Uu_{0_1}$. $\square$

Nonunitary index: Let us now define a numerical characteristic for the field deviation from the unitary field. Let us call the nonunitary index the maximal rank of quadratic forms

$$\sum_{\ell,m=1}^{n} W\left(x_1^{(\ell)}, y_1^{(\ell)}; x_2^{(m)}, y_2^{(m)}\right) Z_\ell \overline{Z}_m, \quad n \le \infty .$$

For the unitary field a nonunitary property coefficient is equal to 0, since $W(x_1,y_1;x_2,y_2) = 0$.

Theorem 2: In order that the field $u(x_1,x_2) = e^{ix_1T_1 + ix_2T_2}u_0$, has a finite nonunitary index it is necessary and sufficiently that dim $H_0 = r < \infty$, where T_1 and T_2 are doubly commuting operators and

$u_0 \in H_0 = \overline{(I - T_1^* T_1)H} \cap \overline{(I - T_2^* T_2)H}$.

Proof:

Sufficiency: When $dim\ H_0 = r < \infty$, there exists representation (5) for $W(x_1, y_1; x_2, y_2)$ and

$$\sum_{\ell,m=1}^{n} W\left(x_1^{(\ell)}, y_1^{(\ell)}; x_2^{(m)}, y_2^{(m)}\right) Z_\ell \overline{Z}_m = \sum_{v=1}^{r} \lambda_v |\zeta_v|^2,$$

where $\zeta_v = \sum_{\ell=1}^{n} \Phi_v\left(x_1^{(\ell)}, x_2^{(\ell)}\right) Z_\ell$. It follows that the rank of quadratic form does not exceed *r*.

Necessity: Let us consider the sequence of pares of real numbers

$$x_\ell = \left(x_1^{(\ell)}, x_2^{(\ell)}\right),\ (\ell = \overline{1,n}).$$

Then

$$\sum_{\ell,m=1}^{n} W(x_\ell, x_m) Z_\ell \overline{Z}_m = \left\langle (I - T_1^* T_1)(I - T_2^* T_2)h, h \right\rangle$$

where $h = \sum_{\ell=1}^{n} Z_\ell u\left(x_1^{(\ell)}, x_2^{(\ell)}\right)$.

Let

$$H_n = \left\{ h : h = \sum_{\ell=1}^{n} Z_\ell u\left(x_1^{(\ell)}, x_2^{(\ell)}\right) \right\},\ H_n \subset H_u.$$

Consider the subspace $G_n = P_n (I - T_1^* T_1)(I - T_2^* T_2) P_n H_u$, where P_n is the projection operator onto subspace H_n. It is obvious that $G_n \subseteq P_n H_0$ and the rank of form $\sum_{\ell,m=1}^{n} W(x_\ell, x_m) Z_\ell \overline{Z}_m$ is equal to $dim\ G_n$. It is evident that $H_1 \subset H_2 \subset \ldots \subset H_n \subset \ldots$ and $\lim_{n\to\infty} P_n = I$, hence rank $W > dim\ G_n$ and

rank $W \geq \lim_{n\to\infty} G_n = dim\ H_0$. This implies that rank $dim\ H_0 \leq r$.

Similarly one may prove the next theorem.

Theorem 3: *In order that the field*

$$u(x_1, x_2) = e^{ix_1 T_1 + ix_2 T_2} u_0,$$

has a finite nonunitary index it is necessary and sufficient that the subspaces

$$H_0^{(j)} = \overline{(I - T_j^* T_j)H} \quad (j = 1, 2)$$

be finite-dimensional where, $u_0 \in H$, T_j are doubly commuting operators.

Further development of suggested approach is related to the spectral theory for the doubly commuting contraction systems and their triangular and universal models[6]. Thus, one may derive canonical representation for $W(x_1, y_1; x_2, y_2)$ and perform harmonic analysis of two-parametric semigroups $e^{ix_1 T_1 + ix_2 T_2}$ when T_1 *and* T_2 are doubly commuting contractions.

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