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Bayes Estimation of the Lifetime Parameters for the Exponential Distribution

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Abstract: Bayes estimation for the parameters of the lifetime distribution when both survival and censoring time are exponentially distributed were studied. The marginal posteriors, Bayes estimates and credible sets were derived using conjugate priors.

Keywords: Bayes estimation; credible sets, survival functions

INTRODUCTION

There are many scenarios in life-testing and reliability experiments in which units are lost or removed from the test before failure. The Bayesian approach requires specifying a probability distribution for the observed data and a prior probability of possible distributions, then all the inference is based on the posterior distribution given the data. Many authors have discussed the Bayesian approach for different scenarios or distributions in both parametric and nonparametric survival data. A Bayes method to infer an unknown failure time distribution method based on the piecewise exponential distribution and a relationship between values of the failure rate in successive intervals presented by Gamerman [3]; it provides smooth estimates of the survival and hazard functions of the distribution. Bayesian survival estimation of the Pareto distribution of the second kind on failure-censored data was proposed by Howlader and Hossian [4], Mostert et al. [5] presented the Bayesian analysis of survival data using the Rayleigh model and linex loss, Mostert et al. [6] introduced the Bayes estimators of the lifetime parameters using the compound Rayleigh model. A nonparametric Bayesian estimation of survival curves from incomplete observations was discussed by Susarla and Van Ryzin [7]. In this paper, a parametric Bayes estimation for the lifetime parameters were discussed when the lifetime and the censoring time are exponentially distributed.

A randomly censored data set consists of n i.i.d pairs (Y_i, D_i), where $Y_i = \text{Min}\,(X_i, T_i)$ and $D_i = I\,(X_i \le T_i)$ for $i=1, 2, \dots n$. In the context of survival analysis and reliability, X_i refers to the survival time and T_i refers to the censoring time. Assume that X_i, the survival time, $i=1, 2, \dots n$, and T_i, censoring time, are independent exponentially distributed with probability density functions given by:

$$f_{X_i}(x_i, \theta) = \frac{1}{\theta} e^{-x_i/\theta}, \quad x_i > 0, \ \theta > 0$$

and

$$f_{T_i}(t_i, \lambda) = \frac{1}{\lambda} e^{-t_i/\lambda}, \quad t_i > 0, \ \lambda > 0$$

respectively. Suppose we observe $[(Y_i, D_i); i=1, 2, n]$, then the probability density function (p.d.f) of D_i is:

$$\begin{aligned} P(D_i = 1) &= P(X_i \le T_i) \\ &= \int_0^\infty \int_{x_i}^\infty \frac{1}{\theta} e^{-x_i/\theta} \frac{1}{\lambda} e^{-t_i/\lambda} dt_i dx_i \\ &= \frac{\lambda}{\lambda + \theta} \end{aligned}$$

and hence,

$$P(D_i = 0) = \frac{\theta}{\lambda + \theta}.$$

It can be shown that the joint probability density function of (Y_i, D_i) is:

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$$f_{Y_i,D_i}(y_i,d_i,\theta,\lambda)=\left(\frac{1}{\theta}+\frac{1}{\lambda}\right)e^{-\left(\frac{1}{\theta}+\frac{1}{\lambda}\right)y_i}\left(\frac{\lambda}{\lambda+\theta}\right)^{d_i}\left(\frac{\theta}{\lambda+\theta}\right)^{1-d_i},\ y_i>0,\ \theta,\ \lambda>0 \qquad (1)$$

and the likelihood function is given by

$$L(\theta,\lambda)=\left(\frac{1}{\theta}+\frac{1}{\lambda}\right)^{n}e^{-\left(\frac{1}{\theta}+\frac{1}{\lambda}\right)\sum_{i=1}^{n}y_i}\left(\frac{\lambda}{\theta}\right)^{\sum_{i=1}^{n}d_i}\left(\frac{\theta}{\lambda+\theta}\right)^{n} \qquad (2)$$

It is shown that the maximum likelihood estimators of θ and λ are given by (See Abu-Taleb and Smadi, [1])

$$\hat{\theta}=\frac{\sum_{i=1}^{n}Y_i}{\sum_{i=1}^{n}D_i}$$

and

$$\hat{\lambda}=\frac{\sum_{i=1}^{n}Y_i}{n-\sum_{i=1}^{n}D_i}$$

BAYES ESTIMATION

We consider the following informative priors on the parameters θ and λ

$$\pi(\theta,\lambda)=\pi_1(\theta)\pi_2(\lambda)$$

$$=\frac{b_1^{a_1}(\frac{1}{\theta})^{a_1-1}e^{-\frac{b_1}{\theta}}}{\Gamma(a_1)}\frac{b_2^{a_2}(\frac{1}{\lambda})^{a_2-1}e^{-\frac{b_2}{\lambda}}}{\Gamma(a_2)},\quad \theta>0,\lambda>0. \qquad (3)$$

This prior assumes the independence of θ and λ with an inverted gamma prior on θ with hyper parameters a_1 and b_1, and an inverted gamma prior on λ with hyper parameters a_2 and b_2.

The likelihood function of θ and λ for fixed **X** and **D** is given by

$$\pi(X,D\mid\theta,\lambda)=\left(\frac{1}{\theta}+\frac{1}{\lambda}\right)^{n}e^{-\left(\frac{1}{\theta}+\frac{1}{\lambda}\right)\sum_{i=1}^{n}y_i}\left(\frac{\lambda}{\theta}\right)^{\sum_{i=1}^{n}d_i}\left(\frac{\theta}{\lambda+\theta}\right)^{n} \qquad (4)$$

Using Bayes theorem, the joint posterior distribution is given by

$$\pi(\theta\ \lambda\mid\quad)=\frac{\pi(\theta\ \lambda)\pi(\quad\mid\theta\ \lambda)}{\int\pi(\theta\ \lambda)\pi(\quad\mid\theta\ \lambda)d\theta d\lambda} \qquad (5)$$

or

$$\pi(\theta,\lambda\mid X,D)\propto\pi(\theta,\lambda)\pi(X,D\mid\theta,\lambda) \qquad (6)$$

Combining the likelihood function given in (4) with the joint prior distribution given in (3) yields:

$$\pi(\theta,\lambda\mid X,D)\propto\frac{b_1^{a_1}(\frac{1}{\theta})^{a_1-1}e^{-\frac{b_1}{\theta}}}{\Gamma(a_1)}\frac{b_2^{a_2}(\frac{1}{\lambda})^{a_2-1}e^{-\frac{b_2}{\lambda}}}{\Gamma(a_2)}\ \text{x}$$

$$\left(\frac{1}{\theta}+\frac{1}{\lambda}\right)^{n}e^{-\left(\frac{1}{\theta}+\frac{1}{\lambda}\right)\sum_{i=1}^{n}y_i}\left(\frac{\lambda}{\theta}\right)^{\sum_{i=1}^{n}d_i}\left(\frac{\theta}{\lambda+\theta}\right)^{n} \qquad (7)$$

after simplification we get

$$\pi(\theta,\lambda\mid y,d)\propto(\frac{1}{\theta})^{\sum_{i=1}^{n}d_i+a_1-1}e^{-\frac{1}{\theta}(b_1+\sum_{i=1}^{n}y_i)}\ \text{x}$$

$$(\frac{1}{\lambda})^{n-\sum_{i=1}^{n}d_i+a_2-1}e^{-\frac{1}{\lambda}(b_2+\sum_{i=1}^{n}y_i)} \qquad (8)$$

The factors involving θ and λ in this last expression are clearly recognizable as inverted gamma posterior distributions. The marginal posterior of θ follows an inverted gamma distribution with parameters $\sum_{i=1}^{n}d_i+a_1$ and $\sum_{i=1}^{n}y_i+b_1$, also, the marginal posterior of λ follows an inverted gamma distribution with parameters $n-\sum_{i=1}^{n}d_i+a_2$ and $\sum_{i=1}^{n}y_i+b_2$; hence the Bayes estimator of θ with respect to squared-error loss function is the mean of the marginal posterior distribution of θ given $\sum_{i=1}^{n}d_i$ and $\sum_{i=1}^{n}y_i$ and is given by

$$\hat{\theta}_B=\frac{\sum_{i=1}^{n}Y_i+b_1}{\sum_{i=1}^{n}D_i+a_1}$$

Similarly, the Bayes estimator of λ is

$$\hat{\lambda}_B=\frac{\sum_{i=1}^{n}Y_i+b_2}{n-\sum_{i=1}^{n}D_i+a_2}$$

As expected, the Bayes estimators updated the information in the observed sample with the information using the prior information specified in the hyper parameters.

To derive the credible sets, if $\pi(\theta \mid x)$ is the posterior distribution of θ given **X**=**x,** then for any set $A \subset \Theta$, the credible probability of *A* is

$$P(\theta \in A \mid x) = \int_A \pi(\theta \mid x) d\theta \quad (9)$$

and *A* is a *credible set* for θ.

Given that the marginal posterior of θ follows an inverted gamma distribution with parameters $\sum_{i=1}^{n} d_i + a_1$ and $\sum_{i=1}^{n} y_i + b_1$, we can form a credible set for θ in many different ways, as any set *A* satisfying (9) will do. One simple way is to split the α equally between the upper and lower endpoints. It follows that

$$2(\sum_{i=1}^{n} y_i + b_1)\theta \sim \chi^2_{2(\sum_{i=1}^{n} d_i + a_1)}$$

Thus a 1-α credible interval is

$$\left\{ \theta : \frac{2(\sum_{i=1}^{n} y_i + b_1)}{\chi^2_{2(n - \sum_{i=1}^{n} d_i + a_1), 1-\frac{\alpha}{2}}} \le \theta \le \frac{2(\sum_{i=1}^{n} y_i + b_1)}{\chi^2_{2(n - \sum_{i=1}^{n} d_i + a_1), \frac{\alpha}{2}}} \right\}.$$

If $a_1 = b_1 = 0$, the posterior distribution of θ given $\sum_{i=1}^{n} d_i$ and $\sum_{i=1}^{n} y_i$ can then be expressed as

$$2(\sum_{i=1}^{n} y_i)\theta \sim \chi^2_{2(\sum_{i=1}^{n} d_i)}.$$

Given that the marginal posterior of λ follows an inverted gamma distribution with parameters $n - \sum_{i=1}^{n} d_i + a_2$ and $\sum_{i=1}^{n} y_i + b_2$, also it follows that (See Casella and Berger, [2])

$$2(\sum_{i=1}^{n} y_i + b_2)\lambda \sim \chi^2_{2(n - \sum_{i=1}^{n} d_i + a_2)}$$

Thus a 1-α credible interval is

$$\left\{ \lambda : \frac{2(\sum_{i=1}^{n} y_i + b_2)}{\chi^2_{2(n - \sum_{i=1}^{n} d_i + a_2), 1-\frac{\alpha}{2}}} \le \lambda \le \frac{2(\sum_{i=1}^{n} y_i + b_2)}{\chi^2_{2(n - \sum_{i=1}^{n} d_i + a_2), \frac{\alpha}{2}}} \right\}.$$

If $a_2 = b_2 = 0$, the posterior distribution of θ given $\sum_{i=1}^{n} d_i$ can then be expressed as

$$2(\sum_{i=1}^{n} y_i)\lambda \sim \chi^2_{2(n - \sum_{i=1}^{n} d_i)}$$

CONCLUSION

Bayes estimation for the parameters of the survival functions when both survival and censoring time are exponentially distributed were studied. The marginal posteriors, Bayes estimates and credible sets were derived using conjugate inverted gamma priors.

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