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Some Stability Results on Krasnolslseskij and Ishikawa Fixed Point Iteration Procedures

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Abstract: In this study, we establish some stability results for the Krasnoselskij and the Ishikawa iteration procedures. We employ the same method as in Berinde[1], but using a more general contractive definition than those of Berinde[1], Rhoades[2], Harder and Hicks[3] and Osilike[4,5].

Key words: Krasnolslseskij, Ishikawa, iteration procedures, contractive definitions

INTRODUCTION

Let (E,d) be a complete metric space and $T:E\rightarrow E$ a selfmap of E . Let $F(T) = \{p \in E \mid T_{p=p}\}$ denote the set of fixed points of T. Let $\{x_n\}_{n=o}^{\infty}$ be the sequence generated by an iteration procedure

$$x_{n+1} = f(T,x_n),\ n = 0, 1, 2,\ldots \qquad (1)$$

Where, $x_o\in E$ is the initial approximation and f is some function. Suppose that $\{x_n\}_{n=o}^{\infty}$ converges to a fixed point p of T.

Let $\{y_n\}_{n=o}^{\infty} \subset E$ and set $\in_n = d(y_{n+1}, f(T,y_n))$, n = 0, 1, 2,…. Then, the iteration procedure (I) is said to be T-stable with respect to T, if and only if, $\lim_{n\to\infty} \in_n = 0$ implies $\lim_{n\to\infty} y_n = p$

Harder and Hicks[3] employed the concept above in proving several stability results under various contractive definitions. Rhoades[2,6] extended the results of Harder and Hicks[3] to other classes of contractive mappings. Specifically, Rhoades[2] extended the results of Harder and Hicks[3] to the following contractive definition: there exists a constant c, $0 \le c < 1$ such that, for each $x,y \in E$,

$$\|T_x-T_y\| \le c \max\{\|x-y\|,\ \frac{1}{2}(\|x-Tx\| + \|y-T_y\|),\ \|x-T_y\|,\ \|y-T_x\|\}. \qquad (2)$$

Using (2), Rhoades[2] established several stability results which are generalizations of the results of Harder and Hicks[3]. It was shown in Rhoades[2] that

$$d(Tx, Ty) \le \frac{c}{1-c} d(x, Tx) + cd(x, y). \qquad (3)$$

Osilike[4] extended the results of Rhoades[2] to the following contractive definition: there exist constants $L \ge 0$, $a\in [0,1)$ such that, for each $x, y \in E$

$$d(Tx,Ty) \le Ld(x,Tx) + ad(x,y). \qquad (4)$$

Osilike[4] proved several stability results using (4). Most of the results of Osilike[4] are generalizations of the results of Rhoades[2] which are themselves generalizations of the results of Harder and Hicks[3]. Berinde[1] using a different method, proved the same results as Harder and Hicks[3] for the same iteration procedures, using the contractive definition (4) above.

In this study, we present some stability results for Krasnoselskij and Ishikawa iteration processes using a more general contractive definition than those of Harder and Hicks[3], Osilike[4], Rhoades[2,6] and Berinde[1]. We shall however employ the method of Berinde[1] in our proofs.

Preliminaries: Let $\{x_n\}_{n=o}^{\infty}$ be the sequence generated by the iteration procedure (1). Then, the Krasnoselskij iteration procedure is obtained from (1) when $f(T,x_n) = \frac{1}{2}(x_n + Tx_n)$, $n \ge 0$, while the Ishikawa iteration process is obtained from (1) when $f(T,x_n) = (1-\alpha_n)x_n + \alpha_n Tz_n$,

$$z_n = (1-\beta_n)x_n + \beta_n Tx_n,\ n \ge 0.$$

We shall employ the following contractive definition: there exist a constant $b \in [0,1)$ and a monotone increasing function $\varphi : \Re_+ \rightarrow \Re_+$ with $\varphi(0) = 0$, such that, for each $x,y \in E$,

$$\|Tx - Ty\| \le \varphi\|x - Tx\| + b\|x - y\|. \qquad (5)$$

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The contractive definition (5) is more general in the following sense: If $\varphi(u) = Lu$, $L \geq 0$ in (5), then we obtain the contractive mapping of Osilike[4]. If $\varphi(u) = \frac{c}{1-c}u$ in (5), then we have the contractive mapping of Rhoades[2]. Also, if $L = 2\delta$ and $a = \delta$ in[3] where $\delta = \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$, $0 \leq \alpha < 1$, $0 \leq \beta < 0.5$, $0 \leq \Upsilon \leq 0.5$, then we obtain the Zamfirescu's contraction in Harder and Hicks[3] and Berinde[1]. When $\varphi(u) = 0$, then (5) reduces to

$$\|Tx-Ty\| \leq b\|x-y\|, \; b \in [0,1), \tag{6}$$

which is a contractive definition in Harder and Hicks[3], Berinde[1] .

In the sequel, we shall require the following Lemma due to Berinde[1].

Lemma 1 (Berinde[1]): If δ is a real number such that $0 \leq \delta < 1$, and $\{\in_n\}_{n=o}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \in_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=o}^{\infty}$ satisfying $u_{n+1} \leq \delta u_n + \in_n$, $n = 0,1\ldots$ we have $\lim_{n\to\infty} u_n = 0$

MAIN RESULTS

We first establish a stability result for the Krasnolseskij iteration procedure as follows.

Theorem 1: Let $\{y_n\}_{n=o}^{\infty} \subset E$ and $\in_n = \|y_{n+1} - \frac{1}{2}(y_n + Ty_n)\|$. Let $(E, \|.\|)$ be a normed linear space and $T: E \to E$ a selfmap of E satisfying (5). Suppose T has a fixed point p. For arbitrary $x_o, \in E$, define sequence $\{x_n\}_{n=o}^{\infty}$ iteratively by; $x_{n+1}=f(T,x_n)=\frac{1}{2}(x_n+Tx_n)$, $n \geq 0$. Let $\varphi:\Re_+\to\Re_+$ be monotone increasing with $\varphi(0)=0$. Then, the Krasnolseskij iteration process is T-stable.

Proof: Let $\lim_{n\to\infty} \in_n = 0$. We shall establish that $\lim_{n\to\infty} y_n = p$. Using (5) and the triangle inequality:

$$\|y_{n+1} - p\| \leq \|y_{n+1} - \tfrac{1}{2}(y_n + Ty_n)\| + \|\tfrac{1}{2}(Ty_n - p)\|$$

$$= \in_n + \tfrac{1}{2}\|(y_n - p) + (Ty_n - p)\|$$

$$\leq \tfrac{1}{2}\{\|y_n - p\| + \|Ty_n - p\|\} + \in_n$$

$$= \tfrac{1}{2}\{\|y_n - p\| + \|p - Ty_n\|\} + \in_n$$

$$= \tfrac{1}{2}\{\|y_n - p\| + \|Tp - Ty_n\|\} + \in_n$$

$$\leq \tfrac{1}{2}\{\|y_n - p\| + \varphi(\|p - Tp\|) + b\|p-y_n\|\} + \in_n$$

$$(\tfrac{1+b}{2})\|y_n - p\| + \in_n \tag{8}$$

Since $0 \leq (\frac{1+b}{2}) < 1$, then by Lemma 1, (8) yields $\lim_{n\to\infty}\|y_n - p\| = 0$.

This implies that $\lim_{n\to\infty} y_n = p$

Conversely, suppose $\lim_{n\to\infty} y_n = p$. Then, from our hypothesis

$$\in_n = \|y_{n+1} - \tfrac{1}{2}(y_n + Ty_n)\|$$

$$\leq \|y_{n+1} - p\| + \|p - \tfrac{1}{2}(y_n + Ty_n)\|$$

$$= \|y_{n+1} - p\| + \|\tfrac{1}{2}(p - y_n + p - Ty_n)\|$$

$$\leq \|y_{n+1} - p\| + \tfrac{1}{2}\|y_n - p\| + \tfrac{1}{2}\|p - Ty_n)\|$$

$$= \|y_{n+1} - p\| + \tfrac{1}{2}\|y_n - p\| + \tfrac{1}{2}\|Tp - Ty_n)\|$$

$$\leq \|y_{n+1} - p\| + \tfrac{1}{2}[\varphi(\|p-Tp\|) + b\|p-y_n\|] + \tfrac{1}{2}\|y_n - p\|$$

$$= \|y_{n+1} - p\| + (\tfrac{1+b}{2})\|y_n - p\| \to 0 \text{ as } n\to\infty.$$

This completes the proof.

Remark 1: By similar argument as above, it is easy to establish a more general case of Theorem 1, which is stated as follows:

Theorem 2: Let $\{y_n\}_{n=o}^{\infty} \subset E$ and $\in_n = \|y_{n+1} - (1-a)y_n - aTy_n\|$, $n \geq 0$. Let $(E, \|.\|)$ be a normed linear space and $T: E \to E$ a selfmap of E satisfying (5). Suppose T has a fixed point p. For arbitrary $x_o, \in E$, define sequence $\{x_n\}_{n=o}^{\infty}$ iteratively by: $x_{n+1} = f(T,x_n) = (1-a)x_n + aTx_n)$, $n \geq 0$, $a \in [0,1]$. Let $\varphi: \Re_+ \to \Re_+$ be monotone increasing and $\varphi(0)=0$. Then, the Schaefer's iteration process(or the Krasnolseskij iteration in the general form) is T-stable.

Remark 2: Specifically, if a = 1 in Theorem 2, we obtain the stability result for the Picard iteration process, (Imoru and Olatinwu[7] and if $a=\frac{1}{2}$ in Theorem above, we obtain Theorem 1.

We now establish stability result for the Ishikawa iteration process.

Theorem 3: Let $\{y_n\}_{n=o}^{\infty} \subset E$ and define $s_n = (1-\beta_n)y_n + \beta_n Ty_n$, $n \geq 0$, let $\in_n = \|y_{n+1} - (1-\alpha)y_n - \alpha Ts_n\|$. Let $(E, \|.\|)$ be a normed linear space and $T: E \to E$ a selfmap of E satisfying (5). Suppose T has a fixed point p and $\varphi: \Re_+ \to \Re_+$ is monotone increasing and $\varphi(0)=0$. For arbitrary $x_o, \in E$, define sequence $\{x_n\}_{n=o}^{\infty}$ iteratively by:

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Tz_n \\ z_n &= (1-\beta_n)x_n + \beta_n Tx_n \end{aligned}\right\} \quad n \geq 0$$

Where, $\{\alpha_n\}_{n=o}^{\infty}$ and $\{\beta_n\}_{n=o}^{\infty}$ are real sequences satisfying:

(i) $\alpha_o = 1$
(ii) $0 \leq \alpha_n, \beta_n \leq 1$, $n \geq 0$ and
(iii) $0 \leq (1-\alpha_n + \alpha_n b - \alpha_n\beta_n b + \alpha_n\beta_n b^2) \leq (1-\alpha_n + \alpha_n b) < 1$.

Then, the Ishikawa iteration process is T-stable.

Proof:

$$\begin{aligned} \|y_{n+1} - p\| &\leq \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\| \\ &\quad + \|1-\alpha_n)y_n - \alpha_n Ts_n - p\| \\ &= \in_n + \|(1-\alpha_n)y_n + \alpha_n Ts_n - [(1-\alpha_n) + \alpha_n]p\| \\ &= \|(1-\alpha_n)(y_n - p) + \alpha_n(Ts_n - p)\| + \in_n \\ &\leq (1-\alpha_n)\|y_n - p\| + \alpha_n\|Ts_n - p\| + \in_n \\ &= (1-\alpha_n)\|y_n - p\| + \alpha_n\|p - Ts_n\| + \in_n \\ &= (1-\alpha_n)\|y_n - p\| + \alpha_n\|Tp - Ts_n\| + \in_n \end{aligned}$$

Since T satisfies (5), we have:

$$\begin{aligned} &\|y_{n+1} - p\| \leq (1-\alpha_n)\|y_n - p\| + \alpha_n[\varphi(p - Tp\|) + b\|p - s_n\|] + \in_n \\ &= (1-\alpha_n)\|y_n - p\| + \alpha_n b(\|(1-\beta_n)(p - y_n) + \beta_n(p - Ty_n)\| + \in_n \\ &\leq (1-\alpha_n)\|y_n - p\| + \alpha_n b(1-\beta_n)\|(p - y_n) + \beta_n(p - Ty_n)\| + \in_n \\ &= (1-\alpha_n + \alpha_n b - \alpha_n\beta_n b)\|y_n - p\| + \alpha_n\beta_n b\|Tp - Ty_n\| + \in_n \\ &= (1 - \alpha_n + \alpha_n b - \alpha_n\beta_n b)\|y_n - p\| + \alpha_n\beta_n b[\varphi(\|p - Tp\|) \\ &\quad + b\|p - y_n\| + \in_n \\ &= (1 - \alpha_n + \alpha_n b - \alpha_n\beta_n b + \alpha_n\beta_n b^2)\|y_n - p\| + \in_n \\ &\leq (1 - \alpha_n + \alpha_n b)\|y_n - p\| + \in_n \end{aligned} \qquad (9)$$

Since $0 \leq (1 - \alpha_n + \alpha_n b) < 1$, then using Lemma 1 in (9) yields $\lim_{n\to\infty}\|y_n - p\| = 0$. This implies that $\lim_{n\to\infty} y_n = p$.

Conversely, let $\lim_{n\to\infty} y_n = p$. Then,

$$\begin{aligned} \in_n &= \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\| \\ &\leq \|y_{n+1} - p\| + \|p - (1-\alpha_n)y_n - \alpha_n Ts_n\| \\ &= \|y_{n+1} - p\| + \|[(1-\alpha_n) + \alpha_n]p - (1-\alpha_n)y_n - \alpha_n Ts_n\| \\ &= \|y_{n+1} - p\| + \|(1-\alpha_n)(p - y_n) + \alpha_n(p - Ts_n\| \\ &\leq \|y_{n+1} - p\| + (1-\alpha_n)\|y_n - p\| + \alpha_n\|p - Ts_n\| \\ &= \|y_{n+1} - p\| + (1-\alpha_n)\|y_n - p\| + \alpha_n\|Tp - Ts_n\| \\ &\leq \|y_{n+1} - p\| + (1-\alpha_n)\|y_n - p\| + \alpha_n[\varphi(\|p - Tp\|) \\ &\quad + b\|p - s_n\|] \\ &= \|y_{n+1} - p\| + (1-\alpha_n)\|y_n - p\| + \alpha_n b\|(1-\beta_n)(p - y_n) \\ &\quad + \beta_n(p - Ty_n)\| \\ &\leq \|y_{n+1} - p\| + (1-\alpha_n)\|y_n - p\| + \alpha_n b\|(1-\beta_n)(p - y_n) \\ &\quad + \alpha_n\beta_n b\|p - Ty_n\| \\ &= \|y_{n+1} - p\| + (1-\alpha_n + \alpha_n b - \alpha_n\beta_n b\|y_n - p\| \\ &\quad + \alpha_n\beta_n b)\|Tp - Ty_n\| \\ &\leq \|y_{n+1} - p\| + (1-\alpha_n + \alpha_n b - \alpha_n\beta_n b\|y_n - p\| \\ &\quad + \alpha_n\beta_n b[\varphi(\|p - Tp\|) + b\|p - y_n\|] \\ &= (1 - \alpha_n + \alpha_n b - \alpha_n\beta_n b + \alpha_n\beta_n b^2)\|y_n - p\| + \in_n \\ &\leq \|y_{n+1} - p\| + (1 - \alpha_n + \alpha_n b)\|y_n - p\| \to 0, \text{ as } n \to \infty. \end{aligned}$$

This completes the proof.

Remark 3: Theorem 3 in this study is a generalization of Theorem 2 of Osilike[4] which is itself a generalization of Theorem 30 of Rhoades[2].

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