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On Concircular Structure Spacetimes

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Abstract: This study presents a study of concircular structure spacetimes which are connected 4-dimensional Lorentzian concircular structure manifolds.

Keywords: $(LCS)_n$-Manifold, Perfect Fluid Spacetime, Scalar Curvature, Killing Vector Field

INTRODUCTION

In general relativity the matter content of the spacetime is described by the energy momentum tensor T which is to be determined from physical considerations dealing with the distribution of matter and energy. Since the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological models, the physical motivation for studying Lorentzian manifolds is the assumption that a gravitational field may be effectively modeled by some Lorentzian metric defined on a suitable four dimensional manifold M. The Einstein equations are fundamental in the construction of cosmological models which imply that the matter determines the geometry of the spacetime and conversely the motion of matter is determined by the metric tensor of the space which is non-flat.

Recently the first author introduced the notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with an example of dimension 4 [4]. The object of the present study is to study the spacetimes which are connected $(LCS)_4$-manifolds and such manifolds will be called concircular structure spacetimes (briefly $(CS)_4$-spacetimes). In this study we investigate the applications of $(LCS)_4$-manifolds to general relativity and physics. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of perfect fluid $(CS)_4$- spacetimes with the characteristic vector field ξ of the spacetime as the flow vector field of the fluid and observed that in such a spacetime the flow vector field of the fluid is irrotational and also the acceleration vector of the fluid must vanish although the expansion scalar of the fluid does not vanish. Also it is shown that if a Ricci-semisymmetric perfect fluid $(CS)_4$- spacetime obeys Einstein equation with cosmological constant then the matter content cannot be a perfect fluid with $\sigma + p \neq 0$, where σ and p are respectively the density and pressure of the fluid. In section 4 we investigate the possibility of a fluid $(CS)_4$- spacetime to admit heat flux and it is proved that in such a spacetime with the characteristic vector field ξ of the spacetime as the flow vector field of the fluid, the matter content can not be a non-thermalised fluid. The last section deals with a Ricci- semisymmetric $(CS)_4$-spacetime and it is shown that in such a spacetime obeying Einstein equation, the characteristic vector field ξ of the spacetime is a Killing vector field if and only if the Lie derivative of the energy momentum tensor with respect to ξ is zero and also we obtain a necessary and sufficient condition for the vector field ξ to be a conformal Killing vector field.

Preliminaries: An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0, 2) such that for each point p $\in M$, the tensor $g_p : T_pM \times T_pM \rightarrow R$ is a non-degenerate inner product of signature $(-, +, \ldots, +)$, where $T_p\, M$ denotes the tangent vector space of M at p and R is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null,spacelike) if it satisfies $g_p(v, v) < 0$ (resp, ≤ 0, $= 0$, > 0) [2]. The category to which a given vector falls is called its causal character.

Let M^n be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \qquad (1)$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X) \qquad (2)$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} \qquad (\alpha \neq 0) \qquad (3)$$

for all vector fields X, Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = \alpha(X) = \rho\eta(X), \qquad (4)$$

ρ being a certain scalar function.
If we put

$$\phi X = \frac{1}{\alpha} \nabla_X \xi \tag{5}$$

then from (3) and (5) we have

$$\phi X = X + \eta(X)\xi, \tag{6}$$

from which it follows that ϕ is a symmetric (1,1) tensor. Thus the Lorentzian manifold M^n together with the unit timelike concircular vector field ξ, its associated 1- form η and (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$- manifold) [4]. In a $(LCS)_n$- manifold, the following relations hold [4]:

a) $\eta(\xi) = -1$,
b) $\phi\xi = 0$,
c) $\eta(\phi X) = 0$,
d) $g(\phi X, \phi Y) = g(X,Y) + \eta(X)\, \eta(Y)$, (7)

$$\eta(R(X,Y)Z) = (\rho - \alpha^2)\, [g(Y,Z)\, \eta(X) - g(X,Z)\, \eta(Y)], \tag{8}$$

$$S(X, \xi) = (n - 1)\, (\rho - \alpha^2)\, \eta(X), \tag{9}$$

$$R(X,Y)\, \xi = (\rho - \alpha^2)\, [\eta(Y)X - \eta(X)Y], \tag{10}$$

for any vector fields X, Y, Z where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold. General relativity flows from Einstein's equation given by

$$S(X,Y) - \frac{r}{2}\, g(X,Y) + \lambda g(X,Y) = kT(X,Y) \tag{11}$$

for all vector fields X, Y where S is the Ricci tensor of the type (0,2), r is the scalar curvature, λ is the cosmological contant, k is the gravitational constant and T is the energy momentum tensor of type (0,2).
The energy momentum tensor T is said to describe a perfect fluid [2] if

$$T(X,Y) = (\sigma + p)A(X)\, A(Y) + pg(X,Y), \tag{12}$$

where σ is the energy density function, p is the isotropic pressure function of the fluid, A is a non-zero 1-form such that $g(X,U) = A(X)$ for all X, U being the flow vector field of the fluid.
In a $(CS)_4$- spacetime by considering the characteristic vector field ξ of the spacetime as the flow vector field of the fluid, the energy momentum tensor takes the form

$$T(X,Y) = (\sigma + p)\, \eta(X)\, \eta(Y) + pg(X,Y). \tag{13}$$

The another form of the energy momentum tensor in a $(CS)_4$- spacetime is given by

$$T(X,Y) = (\sigma + p)\eta(X)\eta(Y) + pg(X,Y) + \eta(X)\omega(Y) + \eta(Y)\omega(X), \tag{14}$$

where $g(X,V) = \omega(X)$ for all X, V being the heat flux vector filed [3] of the fluid orthogonal to ξ i.e., $g(\xi, V) = 0$.
A Lorentzian manifold M^n is said to be Ricci-semisymmetric if its Ricci tensor S of type (0, 2) satisfies the relation
$R(X,Y).\, S = 0$
where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. The notion of Ricci-semisymmetry was first introduced for a Riemannian manifold by Z.I. SzabÓ [5]. We can easily prove the following:

Proposition 1: In a Ricci-semisymmetric $(LCS)_n$-manifold, the Ricci tensor and the scalar curvature are given by

$$S(X,Y) = (n - 1)\, (\rho - \alpha^2)\, g(X,Y), \tag{15}$$

$$r = n\, (n - 1)\, (\rho - \alpha^2). \tag{16}$$

From (15) it follows that a Ricci-semisymmetric $(LCS)_n$-manifold is Einstein and hence its scalar curvature is constant. Consequently (16) implies that $(\rho - \alpha^2)$ is constant.
The above results will be used in the next sections.

Perfect Fluid $(CS)_4$-Spacetimes: Let us consider a perfect fluid $(CS)_4$ -spacetime with the characteristic vector filed ξ of the spacetime as the flow vector filed of the fluid. From (3), it follows that
$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0$,
which yields $g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X) = 0$.
This means that the flow vector field ξ is irrotational. Again from (5) we obtain by virtue of (7(b)) that $\nabla_\xi \xi = 0$. This implies that the integral curves of ξ are geodesics. Hence we can state the following:

Theorem 1: In a perfect fluid $(CS)_4$-spacetime, the flow vector field ξ is irrotational and its integral curves are geodesics.
Again by definition we have

$$div\xi = \sum_{i=1}^{4} \varepsilon_i g\left(\nabla_{e_i} \xi, e_i\right), \text{ where } \varepsilon_i = g(e_i, e_i),$$

$\{e_i\}$, i= 1, 2, 3, 4 is an orthonormal frame filed, '*div*' denotes the divergence.
From (5) we get
$g(\nabla_X \xi, Y) = \alpha g(\varphi X, Y)$
which yields by virtue of (6) that
$g(\nabla_X \xi, Y) = \alpha[g(X,Y) + \eta(X)\, \eta(Y)]$.

This implies that div $\xi = 3\alpha \neq 0$, because $\alpha \neq 0$.
Since div ξ represents the expansion scalar and $\nabla_\xi \xi$ represents the acceleration vector, we can state the following:

Theorem 2: In a perfect fluid $(CS)_4$-spacetime the acceleration vector of the fluid must vanish but the expansion scalar will never vanish.
Using (13) in (11) we obtain

$$S(X,Y) - \frac{r}{2} g(X,Y) + \lambda g(X,Y)$$
$$= k[(\sigma + p)\, \eta(X)\, \eta(Y) + pg(X,Y)]. \tag{17}$$

Putting X = Y = ξ in (17) and then using (9) (for n=4), (1) and (7(a)) we obtain

$$\sigma = \frac{r - 2\lambda - 6(\rho - \alpha^2)}{2k} \tag{18}$$

From (13) we have

$$tr.(T) = 3p - \sigma. \tag{19}$$

Taking an orthonormal frame field and contracting over X and Y in (11) and then using (19) we get

$$r = k(\sigma - 3p) + 4\lambda. \tag{20}$$

In view of (18) we obtain from (20) that

$$p = \frac{6\lambda - r - 6(\rho - \alpha^2)}{6k} \tag{21}$$

Since σ and r are non-constants, from (18) and (21), it follows that σ and p are not constants. Thus we have the following:

Theorem 3: If a perfect fluid $(CS)_4$-spacetime obeys Einstein equation with cosmological constant then none of the pressure and density of the fluid can be a constant.
Again from (18) and (21) we have

$$\sigma + \mathrm{p} = \frac{\mathrm{r} - 12(\rho - \alpha^2)}{3\mathrm{k}} \tag{22}$$

This implies that $\sigma + p \neq 0$. We now suppose that a perfect fluid $(CS)_4$- spacetime is Ricci- semisymmetric. Then from (16) we have

$$r = 12(\rho - \alpha^2). \tag{23}$$

By virtue of (23) and (22) we obtain
$\sigma + p = 0$.
Hence we can state the following:

Theorem 4: If a Ricci-semisymmetric perfect fluid $(CS)_4$-spacetime obeys Einstein equation with cosmological constant, then the matter content cannot be a perfect fluid with $\sigma + p \neq 0$.

Possibility of a Fluid $(CS)_4$-Spacetime to Admit Heat Flux: This section deals with a $(CS)_4$-spacetime in which the matter distribution is a fluid with the characteristic vector field ξ of the spacetime as the flow vector field of the fluid. If possible, suppose that the matter distribution is described by the energy momentum tensor T(X, Y) given by (14). Since V is the heat flux vector filed orthogonal to ξ, we have $g(\xi, V) = \eta(V) = \omega(\xi) = 0$. Hence from (14) we obtain

$$T(X, \xi) = -\sigma\eta(X) - \omega(X). \tag{24}$$

Putting Y = ξ in (11) and then using (9) and (24) we get

$$[3(\rho - \alpha^2) - \frac{r}{2} + \lambda + k\sigma]\, \eta(X) = -k\omega(X). \tag{25}$$

Since $\omega(\xi) = 0$, (25) yields

$$3(\rho - \alpha^2) - \frac{r}{2} + \lambda + k\sigma = 0. \tag{26}$$

By virtue of (25) and (26) we obtain
$k\omega(X) = 0$.
Since $k \neq 0$, it follows that $\omega(X) = 0$ for all X. Hence in a $(CS)_4$-spacetime the matter distribution cannot be described by the energy momentum tensor of the form (14). This leads to the following :

Theorem 5: If in a $(CS)_4$-spacetime the matter distribution is a fluid with the characteristic vector field ξ of the spacetime as the flow vector field of the fluid, then such a fluid can not admit heat flux.

Ricci-Semisymmetric $(CS)_4$-Spacetimes: In this section we consider a Ricci-semisymmetric $(CS)_4$-spacetime. Then from (15) and (16) we get

$$S(X, Y) = \frac{r}{4} g(X, Y). \tag{27}$$

Using (27) in (11) we have

$$(\lambda - \frac{r}{4})\, g(X, Y) = kT(X, Y). \tag{28}$$

Since in a Ricci-semisymmetric $(CS)_4$-spacetime, r is constant, the relation (28) yields

$$(\lambda - \frac{r}{4})(\pounds_{\xi} g)(X,Y) = k(\pounds_{\xi} T)(X,Y), \quad (29)$$

where $\pounds_{\xi}$ denotes the Lie derivative with respect to ξ. Now if the vector field ξ is a Killing vector field, then $(\pounds_{\xi} g)(X,Y) = 0$ and hence (29) implies that (since $k \neq 0$)

$$(\pounds_{\xi} T)(X,Y) = 0 \quad (30)$$

Conversely, if (30) holds then (29) yields either $\lambda = \frac{r}{4}$,

or $(\pounds_{\xi} g)(X,Y) = 0$ for all X,Y.

But if $\lambda = \frac{r}{4}$, then (28) implies that $T(X,Y) = 0$ for all X,Y which is not possible.

Hence we must have $(\pounds_{\xi} g)(X,Y) = 0$.

Thus we can state the following:

Theorem 6: In a Ricci-semisymmetric $(CS)_4$-spacetime obeying Einstein equation, the characteristic vector field ξ of the spacetime is a Killing vector field if and only if the Lie derivative of the energy momentum tensor with respect to ξ is zero.

Next, if ξ is a conformal Killing vector field, then

$$(\pounds_{\xi} g)(X,Y) = 2\gamma g(X,Y), \quad (31)$$

where γ is a scalar function. Using (31) in (29) we get

$$2(\lambda - \frac{r}{4})\gamma g(X,Y), = k(\pounds_{\xi} T)(X,Y),$$

which yields by virtue of (28) that

$$(\pounds_{\xi} T)(X,Y) = 2\gamma T(X,Y). \quad (32)$$

From (32) we can say that the energy momentum tensor has Lie inheritance property along ξ [1]. Conversely, if (32) holds then it follows that (31) holds and hence ξ is a conformal Killing vector field.

This leads to the following:

Theorem 7: In a Ricci-semisymmetric $(CS)_4$-spacetime obeying Einstein equation, the characteristic vector field ξ of the spacetime is a conformal Killing vector field if and only if the energy momentum tensor has the Lie inheritance property.

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