

Asymptotic Stability of Unicycle-Like Robots: The Bessel's Controller

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Abstract: Asymptotic stability of unicycle-like robots proved to be involved due to Brockett's condition. By using a smooth, time-invariant controller constructed out of Bessel's functions, in this paper unicycle-like robots are uniformly-exponentially stabilized to the origin. The pure feedback controller obtained provides closed-form trajectories with the possibility of a simple and feasible (hardware) non-linear observer construction from posture angle measurements solely. Two examples are presented: Asymptotic steering of a unicycle to the origin using gyros and a perfect non-linear observer reconstruction states along with conclusions and future work.

Keywords: Nonholonomic Dynamics, Brockett's Condition, Kinematic Model, Closed-Loop, Closed-Form Solution

Introduction

Modelling mechanical systems can be carried out in two main ways (Angeles and Kecskemethy, 1995):

- Dynamic models including forces and torques
- Kinematic models excluding forces and torques

Both approaches aim to collect a system of Ordinary Differential Equations (ODE's) parameterized in its control inputs (Astolfi *et al.*, 1997; Bloch, 2015).

These ODE systems are mainly non-linear, with a universal modeling given by Sarker *et al.* (1994) in the case of kinematic rolling constraints.

It turns out that dynamic models represent the most general approach to account for all possible physical interactions that may occur. However, these models are out of as many ODE's as the mechanical systems' degrees of freedom (Kane and Levinson, 1985; Angeles and Kecskemethy, 1995).

At this point, two main challenges must be dealt with:

- A great amount of ODE's
- A control law rendering the system asymptotically stable/stable (Kostić *et al.*, 2009; Udwadia and Kalaba, 1994; Skowronski, 2012)

For these reasons, many researchers focus the attention on more tractable models, yet keeping the non-linear richness with fewer amounts of ODE's (Muñoz-Lecanda and Yániz Fernandez, 2008).

This explains the great interest in controlling kinematic models (Siegwart *et al.*, 2011) with the case of mobile robots as a subset of kinematic modeling, mainly, wheel planar kinematic models.

Moreover, it happens that these models can be classified into two universal classes: Holonomic and Nonholonomic (García and Agamennoni (2012) for a universal classification and models).

As pointed out by Jian *et al.* (2010), El-Hawwary and Maggiore (2008) and Qu *et al.* (2004), holonomic robots are simpler than nonholonomic, being one of the reasons that many techniques have been proposed without any convergence's guarantee (Lavalle, 2006; Galceran and Carreras, 2013; Kumar and Dewangan, 2016; Yang *et al.*, 2016; Salaris *et al.*, 2010; Wang *et al.*, 2009; Thomas *et al.*, 2016).

On the other hand, it is well-known that a nonholonomic robot cannot be stabilized asymptotically with a smooth controller due to Brockett's condition (Brockett, 1983).

For this reason, many different techniques have been proposed to control nonholonomic robots avoiding the use of time-invariant controllers (Zambelli *et al.* (2015) and the references therein).

However, none of the available techniques considers the stabilization's problem in closed-form. In fact, according to Lizárraga (2004), it is not possible to track some desired trajectories with an equi-continuous control law.

To summarize the literature's drawbacks in controlling nonholonomic robots, either path-following or asymptotic stability:

- Planar curves must be parameterized to be followed $f(x,y) = 0$ (Morro *et al.* (2011))
- Path's curvature must satisfy some specifications: $\nabla f(x,y) \neq 0$ (Morro *et al.* (2011))
- Very oscillatory and slow motion (see for instance Moon Kim and Tsiotras (2002))
- Brockett's condition
- Lizárraga's obstructions
- Closed-form solution's unavailability for a universal set of models
- Closed-form algorithms to track/follow any desired pre-specified trajectory

In this paper, generalizing the solution presented in Garcia *et al.* (2008), a smooth time-invariant controller is presented to steer in closed-loop a unicycle-like robot to the origin uniformly exponentially stable.

The contributions in this paper are as follows:

- Closed-form solution to steer unicycle robots to the origin
- Continuous feedback controller with guaranteed stability
- The proof that a unicycle can be asymptotically stabilized measuring considering modularity
- A non-linear observer with angle output measurement (gyros)

This paper is organized as follows: Section *Rolling constraints* presents the modeling of rolling constraints to be considered, trajectories' closed-form solution using a smooth, time-invariant controller, Section *Unicycle's asymptotic stability* presents the asymptotic stability analysis, Section *Practical controller: Only gyros* presents a practical algorithm, whereas Section *Examples* simulates in Matlab. Finally, Section *Conclusions* depicts some conclusions and future work.

Notations and Definitions

In this short section, some definitions are provided to use all along the paper:

Rotation Matrix

$$R(x) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

Matrix Eigenvalues

Given a matrix $A \in \mathbb{R}^{n \times n}$, the eigenvalues $\lambda(A)$:

$$\lambda(A) = \det(\lambda \cdot I - A) = 0$$

With I the identity matrix.

Matrix Transpose

Given a matrix $A \in \mathbb{R}^{n \times n}$, the transpose is denoted by A' .

Derivatives with Respect to Time

Time derivatives are indicated as:

$$\dot{x}(t) = \frac{dx(t)}{dt}$$

Rolling Constraints: Unicycle-Like Robots

As mentioned previously, mechanical systems that roll without slipping encompass the modeling for many mechanical systems (Bloch, 2015).

Moreover, according to Murray and Shankar (1993) any nonholonomic system can be written in a universal chain form, so unicycle models can be considered as a general nonholonomic dynamics.

In particular, unicycle-like robots represents a universal modeling for a wide variety of wheeled robots (Garcia and Agamennoni, 2012) and Fig. 1:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\theta}(t) \end{bmatrix} = \begin{bmatrix} \cos(x) & 0 \\ \sin(x) & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (1)$$

where the control inputs $(u_1, u_2) \in \mathbb{R}^1 \times \mathbb{R}^1$.

Bessel's Functions Closed-form Solutions

Following the ideas in Garcia *et al.* (2008), a lemma can be proved.

Lemma 1

Given the dynamics in Equation 1 driven by the controller:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^N (2 \cdot i + 1) \cdot a \cdot C_i \cdot J_i(\theta) \cdot \theta^{i+1} \\ a \cdot \theta \end{bmatrix}$$

$$\theta(0) = \begin{cases} \theta(0), & \theta(0) \neq 0 \\ 2 \cdot \pi, & \theta(0) = 0 \end{cases}$$

For any arbitrary $N \in \mathbb{N}$, with $a < 0$. The robot's trajectories are given by:

$$\begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} R(\theta) \cdot \sum_{i=1}^N C_i \cdot \begin{bmatrix} J_i(\theta) \cdot \theta^{i+1} \\ -J_{i+1}(\theta) \cdot \theta^{i+1} \end{bmatrix} \\ e^{a \cdot t} \cdot \theta(0) \end{bmatrix} \quad (2)$$

With J_i the Bessel's functions of first kind and C_i arbitrary constants depending on the initial conditions.

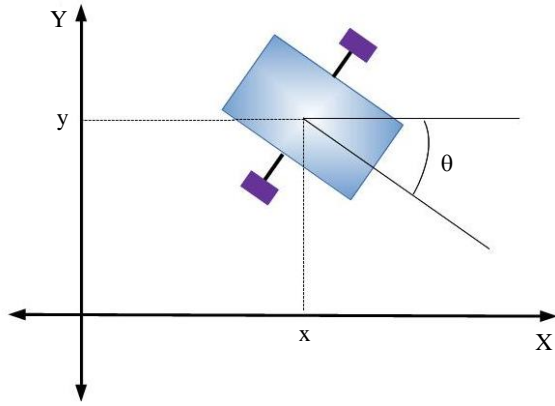


Fig. 1: Unicycle-like robot with coordinates

Proof

The proof is in the Appendix.

Equation 2 states clearly the origin as an equilibrium point if and only if $\lim_{t \rightarrow \infty} \theta(t) = 0$.

Unicycle's Asymptotic Stability

Equation 1 defines a dynamics that it is endowed with uniform exponential stability by using Lemma 1 (Rugh (1995) for a definition on stability).

Theorem 1

The controller in Lemma 1 possess uniform exponentially stability to the origin.

Proof

The proof is in the Appendix.

Clearly, it is a closed-loop and time-invariant controller, except for $\theta(0) = 0$, (rendering the controller identically zero).

However, the posture angle is a modular quantity:

$$\theta(0) = 0 \Leftrightarrow \theta(0) = 2 \cdot \pi \text{ mod } 2 \cdot \pi$$

This modularity property avoids Brockett's condition collision. On the other hand, in a paper by Aicardi *et al.* (1995) a closed-loop controller using Lyapunov techniques was presented, however that controller becomes singular at $x(0) = 0, y(0) = 0$, whereas the controller in this paper is well defined for the whole \mathfrak{R}^3 .

Regular Embedded Sub-Manifold: Trajectories' First Integral

Notwithstanding that the closed-form solution obtained represents the complete system's time evolution; a geometrical point of view is of interest in what follows.

Considering the closed-form trajectories in Lemma 1 with $N = 2$:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = R(\theta)' \cdot \left(C_1 \cdot \begin{bmatrix} J_1(\theta) \cdot \theta^2 \\ -J_2(\theta) \cdot \theta^2 \end{bmatrix} + C_2 \cdot \begin{bmatrix} J_2(\theta) \cdot \theta^3 \\ -J_3(\theta) \cdot \theta^3 \end{bmatrix} \right)$$

Compactly:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = R(\theta)' \cdot L(\theta) \cdot \bar{C}$$

Where:

$$L(\theta) = \begin{bmatrix} J_1(\theta) \cdot \theta^2 & J_2(\theta) \cdot \theta^3 \\ -J_2(\theta) \cdot \theta^2 & -J_3(\theta) \cdot \theta^3 \end{bmatrix}, \bar{C} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

Matrix $L(\theta)$ is nonsingular in the view of the Bourget's hypothesis proved in 1929 by Siegel (Watson, 1966).

Then, trajectories' regular embedded sub-manifold (first integral) follows (Bloch (2015), Isidori (1995) and Nijmeijer and van der Schaft (1990) for details in geometrical control):

$$\bar{C} = L^{-1}(\theta) \cdot R(\theta) \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{3}$$

Equivalently:

$$L^{-1}(\theta) \cdot R(\theta) \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = L^{-1}(\theta(0)) \cdot R(\theta(0)) \cdot \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

Non-Linear Observer

Equation 3 can be utilized to derive a non-linear observer measuring only the angle posture (Luenberger (1966) and Isidori (1995) for linear and non-linear observers):

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = R(\theta)' \cdot L(\theta) \cdot L^{-1}(\theta(0)) \cdot R(\theta(0)) \cdot \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \tag{4}$$

Practical Controller: Only Gyros

Once that uniform exponential stability has been proved in Theorem 1, a practical algorithm can be described to control in closed-loop a unicycle robot:

- Determine the constant vector \bar{C} from Equation 3 given the initial conditions

$$\bar{C} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = L^{-1}(\theta(0)) \cdot R(\theta(0)) \cdot \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

- Use the controller in Lemma 1 with $N=2$ and single gyro measurements:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a \cdot [3 \cdot C_1 \cdot J_1(\theta) \cdot \theta^2 + C_2 \cdot J_2(\theta) \cdot \theta^3] \\ a \cdot \theta \end{bmatrix}$$

$$\theta(0) = \begin{cases} \theta(0), & \theta(0) \neq 0 \\ 2 \cdot \pi, & \theta(0) = 0 \end{cases}$$

Notice that only the initial condition must be provided to initiate the algorithm, endowing the controller with a very strong property for the well-known SLAM problem (Lavalle, 2006).

Only SLAM must be made at $t = 0$ as opposed to the available literature where SLAM or time-tracking has to be performed on-line.

Examples

Using Matlab, Lemma 1 is implemented with both objectives: steering to the origin asymptotically stable and non-linear observer reconstruction.

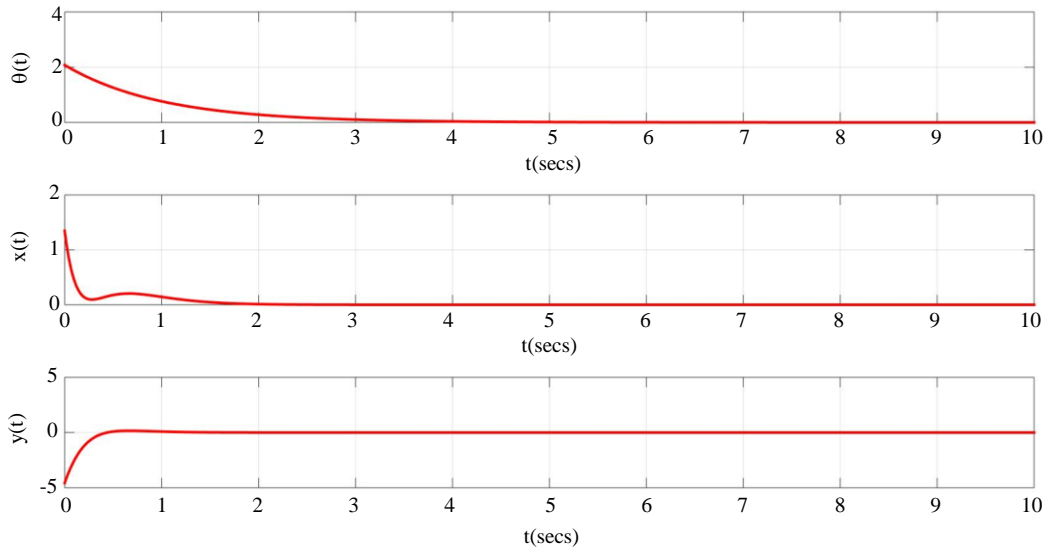


Fig. 2: Unicycle's closed-loop trajectories

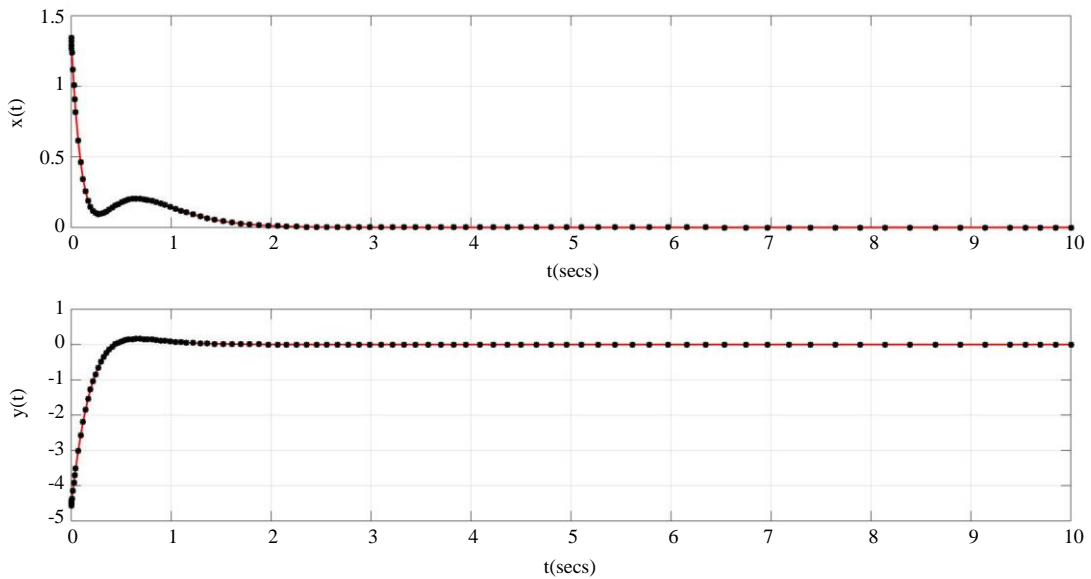


Fig. 3: Numerical verification

Using Gyros

Considering the arbitrary initial condition: [1.345,-4.567,2.078]'. Then, Fig. 2 is obtained.

Non-Linear Observer Verification

Equation 4 provides an interesting verification to numerically reconstruct robot's states. Then, Fig. 3 is obtained.

Discussion

Unicycle robot's asymptotic stability is not possible using a time-invariant feedback controller due to Brockett's condition.

In this paper, the modularity of the angular posture along with a novel pure feedback and time-invariant controller, allows asymptotic stability for unicycle robots.

It should be clear that modularity was not studied previously in the literature, excluding also closed form solutions addressed in this paper.

It turns out that, besides the fact of providing the important concept of a modular controller, SLAM and non-linear observers can be constructed in hardware using posture angle solely.

Conclusion

Nonholonomic trajectories of a unicycle-like robot is solved in closed-form using a smooth, closed-loop and time-invariant controller.

Uniform exponential stability was proved, even for this case (unicycle robot) where Brockett's condition is satisfied on the basis of modularity. In fact, postural angle's modularity was the cornerstone to avoid obstructions using smooth, closed-loop, time-invariant control laws.

Besides the wide variety of available literature, trajectories' closed-form knowledge allowing an explicit non-linear observer derivation to completely reconstruct states using only a single gyro sensor, makes a salient property of this paper.

Possible future work encompasses:

- Numerical algorithm to drive a set of multiple robots with additional constraints (formation of robots)
- Non-linear observer's numerical analysis robustness
- Real-time applications using on-board gyros and microprocessors
- Satellite control using gyros, applying the universal transformation in Murray and Shankar (1993)
- Optimal control

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Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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Appendix

Lemma 1 Proof

Let's consider an auxiliary system:

$$\dot{X}_i = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \cdot u_i, \quad i = 1, \dots, N$$

Summing up:

$$\sum_{i=1}^N \dot{X}_i = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \cdot \sum_{i=1}^N u_i$$

Defining:

$$X = \sum_{i=1}^N X_i \Rightarrow u_1 = \sum_{i=1}^N u_i$$

Considering the inspiring guess:

$$u_i = a \cdot (2 \cdot i + 1) \cdot [\cos(\theta) \quad \sin(\theta)] \cdot X_i \quad (5)$$

$a < 0, \quad i = 1, \dots, N$

Then:

$$\dot{X}_i = a \cdot (2 \cdot i + 1) \cdot R(\theta)' \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot [1 \quad 0] \cdot R(\theta)$$

Defining the change of coordinates:

$$Z_i = R(\theta) \cdot X_i$$

Taking derivatives with respect to time:

$$\dot{Z}_i = \begin{bmatrix} (2 \cdot i + 1) \cdot a & \dot{\theta} \\ -\dot{\theta} & 0 \end{bmatrix} \cdot Z_i \quad (6)$$

Defining the control input:

$$u_2 = a \cdot \theta, \quad a < 0$$

$$\theta(0) = \begin{cases} \theta(0), & \theta(0) \neq 0 \\ 2 \cdot \pi, & \theta(0) = 0 \end{cases}$$

Equation 6 leads:

$$\theta \cdot \frac{dZ_i}{d\theta} = \begin{bmatrix} (2 \cdot i + 1) & \theta \\ -\theta & 0 \end{bmatrix} \cdot Z_i \quad (7)$$

Equation 5 means:

$$u_i = a \cdot (2 \cdot i + 1) \cdot [1 \quad 0] \cdot Z_i$$

So, only the first component of vector Z_i needs to be considered. Taking derivative with respect to θ in Equation 7:

$$\theta^2 \cdot \frac{dZ_{i1}(\theta)}{d\theta} = \theta \cdot (2 \cdot i + 1) \cdot \frac{dZ_{i1}(\theta)}{d\theta} + (2 \cdot i + 1 + \theta^2) \cdot Z_{i1}(\theta)$$

For this Bessel's ODE, a closed-form solution is known (Andrews (1985) pp.229):

$$Z_{i1}(\theta) = \theta^{i+1} \cdot (C_i \cdot J_i(\theta) + D_i \cdot Y_i(\theta))$$

With $\{J_i, Y_i\}$ the Bessel's functions of first and second kind respectively. Assuming $D_i = 0$. Finally, for Z_{i2} from Equation (6):

$$\begin{cases} Z_{i2}(\theta) = \frac{dZ_{i1}(\theta)}{d\theta} - \frac{(2 \cdot i + 1)}{\theta} \cdot Z_{i1}(\theta) \\ Z_{i1}(\theta) = \theta^{i+1} \cdot C_i \cdot J_i(\theta) \end{cases}$$

Utilizing the Bessel's function property:

$$\frac{dJ_i(\theta)}{d\theta} = \frac{i}{\theta} \cdot J_i(\theta) - J_{i+1}(\theta)$$

Then:

$$Z_{i2}(\theta) = C_i \cdot \left[\frac{i}{\theta} \cdot J_i(\theta) \cdot \theta^{i+1} - J_i(\theta) \cdot \theta^{i+1} + (i+1) \cdot J_i(\theta) \cdot \theta^i - \frac{(2i+1)}{\theta} \cdot J_i(\theta) \cdot \theta^{i+1} \right]$$

Equivalently:

$$Z_{i2}(\theta) = C_i \cdot [-\theta^{i+1} \cdot J_{i+1}(\theta)]$$

This completes the proof.

Theorem 1 Proof

First, the origin must be an equilibrium point. Equation 2 shows that the origin is an equilibrium point as long as $\theta(t)$ is exponentially tending to zero.

Moreover, this equilibrium point is attractive to the origin uniformly (taking into account the Bessel's functions of first kind's uniform decay behavior):

$$\dot{\theta}(t) = a \cdot \theta(t), \quad a < 0 \Rightarrow \lim_{t \rightarrow \infty} \theta(t) = 0 \quad (\text{Uniformly})$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = R(\theta) \cdot \sum_{i=1}^N C_i \cdot \theta^{i+1} \cdot \begin{bmatrix} J_i(\theta) \\ -J_{i+1}(\theta) \end{bmatrix} \rightarrow 0 \quad (\text{Uniformly})$$

Finally, an exponential bound is proved:

$$\left\| \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \right\| \leq \|R(\theta)\| \cdot \sum_{i=1}^N C_i \cdot |\theta^{i+1}| \cdot \left\| \begin{bmatrix} J_i(\theta) \\ -J_{i+1}(\theta) \end{bmatrix} \right\|$$

Bessel's functions are bounded at the origin ($\theta = 0$), so uniform exponentially stability is proved. This completes the proof.