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Vibration Equations of Thick Rectangular Plates Using Mindlin Plate Theory

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Abstract: Problem statement: Rectangular steel plates are widely used in various steel structures and steel industries. For a proper design of steel plate structures and efficient use of material, the behavior, strength, buckling and post-buckling characteristics of plates should be accurately determined. **Approach:** Considering the significance of this matter, lateral vibration of thick rectangular plates was studied on the basis of mindlin plate theory. The exact characteristic equations for a plate which is single supported in two opposite edges are available in the literature. S-C-S-F boundary condition which covers all possible situations is selected in this study. **Results:** The plate frequencies were calculated for this boundary condition for a wide range of plate sizes and thicknesses. The plate mode shapes were obtained for different cases and the effect of changes in boundary conditions; size ratio and thickness on the vibration behavior of rectangular steel plates are studied. **Conclusion/Recommendations:** Since the results of this study is exact and without any approximation, the presented values can be used as a proper criteria to evaluate the error value of approximate methods which are used by engineers for design of steel plates. These results can provide a good gridline for efficient design and prevention of using high safety factors. Considering the wide range of steel rectangular plates, more sizes and thicknesses of plates can be studied. The behavior of plates with other boundary conditions can also be studied for future research.

Key words: Mindlin plate theory, vibration, thick plate, mode shape

INTRODUCTION

The Classic Plate Theory (CPT) provides a theoretical model of plate behavior which has some considerable advantages, which cab be employed with confidence over a reasonable range of applications, but which also has significant limitations. The popularity of CPT arises from the fact that the bending behavior of a plate is expressed in terms of a sole, fundamental reference quantity that is w, the lateral displacement of the middle surface. The Kirchhoff hypothesis is used in CPT that straight lines originally normal to the plate middle surface remain straight and normal during the deformation process. The consequence of using this hypothesis is that the effect of through-thickness shear deformation is ignored in CPT and thus the classical theory overestimates the stiffness of the plate. Such overestimation is of little consequence for truly thin plates but can be of very considerable significance for other plates, particularly in vibration and buckling problems when the ratio of plate thickness to typical wavelength increases. A number of plate theories exist in which the Kirchhoff hypothesis is relaxed o take account of shear deformation and related effects and amongst these theories, those of Mindlin and Reissner are closely related and are well known.

The basic assumption of Mindlin plate theory is that a straight line originally normal to the plate middle surface is constrained to remain straight but not generally normal to the middle surface after deformations.

The inclusion of shear deformation effects in Mindlin plate theory means that the two cross-sectional rotations ψ_x and ψ_y have to be considered as independent, fundamental reference quantities, in addition to w. Thus, three fundamental quantities are involved in Mindlin plate theory, against the one of CPT[1].

The assumption of Mindlin plate theory implies that shear strain distribution through the plate thickness are uniform, but this cannot be so. To correct for this, one shear coefficient factor is introduced into the analysis and selection of these factors is of some significance[2].

The present study is to determine the exact characteristics equations for the case of S-C-S-F. Considering the transverse shear deformation, Mindlin plate theory is used to derive the integrated equations of motion in terms of the stress resultant.

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The frequency parameters which are calculated using the exact characteristic equations are obtained for this case, which can cover a wide range of plate aspect ratios η and relative thickness ratio δ. For the mentioned boundary condition S-C-S-F, Three dimensional mode shapes and their contour plots for η = 2 and δ = 0.1 are shown.

MATERIALS AND MATHODS

All the formulations provide here is for a rectangular plate of length a, width b and uniform thickness of h. Such a plate is shown in Fig. 1.

The displacements along the x1 and x_2 axes are respectively marked as U_1 and U_2 and the displacement in the direction perpendicular to plane of x_1 and x_2 is marked as U_3. According to Mindlin plate theory, the value of displacement components in these directions can be calculated by formulas 1:

$$\begin{aligned} U_1 &= -x_3\psi_1(x_1,x_2,t) \\ U_2 &= -x_3\psi_2(x_1,x_2,t) \\ U_3 &= \psi_3(x_1,x_2,t) \end{aligned} \tag{1}$$

Where ψ_1 and ψ_2 are the slope due to bending alone in the respective planes, ψ_3 is the transverse displacement and t is the time. The strains in the form of tensor components can be derived from equation 1 and can be written as Eq. 2:

$$\begin{aligned} \varepsilon_{11} &= -x_3\psi_{1,1} \\ \varepsilon_{22} &= -x_3\psi_{2,2} \\ \varepsilon_{33} &= 0 \\ \varepsilon_{12} &= -\frac{1}{2}(\psi_{1,2}+\psi_{2,1})x_3 \\ \varepsilon_{13} &= -\frac{1}{2}(\psi_1-\psi_{3,1}) \\ \varepsilon_{23} &= -\frac{1}{2}(\psi_2-\psi_{3,2}) \end{aligned} \tag{2}$$

If M_{11} and M_{22} and M_{12} are the bending and twisting moments per unit length and Q_1 and Q_2 are the shear forces per unit length, then the plate linear constitutive relationships can be expressed as Eq. 3:

$$\begin{aligned} M_{11} &= -D(\psi_{1,1}+\upsilon\psi_{2,2}) \\ M_{22} &= -D(\psi_{2,2}+\upsilon\psi_{1,1}) \\ M_{12} &= -\frac{D}{2}(1-\upsilon)(\psi_{1,2}+\psi_{2,3}) \\ Q_1 &= -k^2Gh(\psi_1-\psi_{3,1}) \\ Q_2 &= -K^2Gh(\psi_2-\psi_{3,2}) \end{aligned} \tag{3}$$

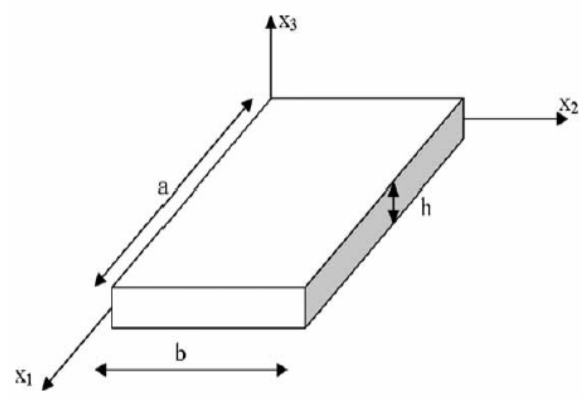

Fig. 1: A Mindlin plate with coordinate convention[3]

Where $D = Eh^3/12(1-\upsilon^2)$, ν as Poisson's ratio and E and G as the modulus of elasticity and rigidity. The constant k^2 is the shear correction factor introduce to account for the non-uniformity of shear strain through the plate thickness.

The equations of motion can be derived from three dimensional equations of motion in the form of Eq. 4:

$$\begin{aligned} \sigma_{11,1}+\sigma_{12,2}+\sigma_{13,3} &= \rho \overset{**}{U}_1 \\ \sigma_{21,1}+\sigma_{22,2}+\sigma_{23,3} &= \rho \overset{**}{U}_2 \\ \sigma_{31,1}+\sigma_{32,2}+\sigma_{33,3} &= \rho \overset{**}{U}_3 \end{aligned} \tag{4}$$

where ρ is mass density per unit volume. Since there is no shear force in the faces of the plate, the integration through the thickness of plate for equations 4 gives Eq. 5:

$$\begin{aligned} M_{11,1}+M_{12,2}-Q_1 &= \frac{1}{12}\rho h^3\omega^2\psi_1 \\ M_{12,1}+M_{22,2}-Q_2 &= \frac{1}{12}\rho h^3\omega^2\psi_2 \\ Q_{1,1}+Q_{2,2} &= -\rho h\omega^2\psi_3 \end{aligned} \tag{5}$$

If the coordinates are normalized to the plate planar dimensions, non-dimensional parameters can be calculated by Eq. 6:

$$\begin{aligned} & X_1 = \frac{x_1}{a},\ X_2 = \frac{x_2}{b},\ \delta = \frac{h}{a} \\ & \eta = \frac{a}{b} \\ & \beta = \omega a^2\sqrt{\frac{\rho h}{D}} \end{aligned} \tag{6}$$

where β is frequency parameter. Equation 3 can now be written in dimensionless form as Eq. 7:

$$\begin{aligned}
&\tilde{M}_{11} = -(\tilde{\psi}_{1,1} + \upsilon\eta\tilde{\psi}_{2,2})e^{i\omega t} = \frac{M_{11}}{D}a \\
&\tilde{M}_{22} = -(\eta\tilde{\psi}_{1,1} + \upsilon\tilde{\psi}_{1,1})e^{i\omega t} = \frac{M_{22}}{D}a \\
&\tilde{M}_{12} = -(\frac{1-\upsilon}{2})(\eta\tilde{\psi}_{1,2} + \tilde{\psi}_{2,1})e^{i\omega t} = \frac{M_{12}}{D}a \\
&\tilde{Q}_{1} = -(\tilde{\psi}_{1} - \tilde{\psi}_{3,1})e^{i\omega t} = \frac{Q_1}{K^2Gh} \\
&\tilde{Q}_{2} = -(\tilde{\psi}_{2} - \eta\tilde{\psi}_{3,2})e^{i\omega t} = \frac{Q_2}{K^2Gh}
\end{aligned} \tag{7}$$

In these equations, partial differentiation with respect to the normalized coordinates is represented by comma subscript. The parameters $\tilde{\psi}_1$, $\tilde{\psi}_2$ and $\tilde{\psi}_3$ can be given by Eq. 8:

$$\begin{aligned}
&\tilde{\psi}_1(X_1,X_2) = \psi_1(x_1,x_2,t)e^{-i\omega t} \\
&\tilde{\psi}_2(X_1,X_2) = \psi_2(x_1,x_2,t)e^{-i\omega t} \\
&\tilde{\psi}_3(X_1,X_2) = \psi_3(x_1,x_2,t)\frac{e^{-i\omega t}}{a}
\end{aligned} \tag{8}$$

If the dimensionless stress resultants of Eq. 7 are substituted in Eq. 5, 9 can be derived:

$$\begin{aligned}
&\tilde{\psi}_{1,11} + \eta^2\tilde{\psi}_{1,22} + \frac{1-\upsilon}{1+\upsilon}\eta(\tilde{\psi}_{1,11} + \eta\tilde{\psi}_{2,12}) - \\
&\frac{12K^2}{\delta^2}(\tilde{\psi}_1 - \eta\tilde{\psi}_{3,1}) = -\frac{\beta^2\delta^2}{6(1-\upsilon)}\tilde{\psi}_1 \\
&\tilde{\psi}_{2,11} + \eta^2\tilde{\psi}_{2,22} + \frac{1-\upsilon}{1+\upsilon}\eta(\tilde{\psi}_{1,12} + \eta\tilde{\psi}_{2,22}) - \\
&\frac{12K^2}{\delta^2}(\tilde{\psi}_2 - \eta\tilde{\psi}_{3,2}) = -\frac{\beta^2\delta^2}{6(1-\upsilon)}\tilde{\psi}_2 \\
&\tilde{\psi}_{3,11} + \eta^2\tilde{\psi}_{3,22} - (\tilde{\psi}_{1,1} + \eta\tilde{\psi}_{2,2}) = \\
&-\frac{\beta^2\delta^2}{6(1-\upsilon)}\tilde{\psi}_3
\end{aligned} \tag{9}$$

These equations can be solved if the functions $\tilde{\psi}_1$, $\tilde{\psi}_2$ and $\tilde{\psi}_3$ are written in the form of three dimensionless potentials W1, W2 and W3 as Eq. 10:

$$\begin{aligned}
&\tilde{\psi}_1 = (1 - \frac{2\alpha_2^2}{(1-\upsilon)\alpha_3^2})W_{1,1} + (1 - \frac{2\alpha_1^2}{(1-\upsilon)\alpha_3^2})W_{2,1} - \eta W_{3,2} \\
&\tilde{\psi}_2 = (1 - \frac{2\alpha_2^2}{(1-\upsilon)\alpha_3^2})W_{1,2} + (1 - \frac{2\alpha_1^2}{(1-\upsilon)\alpha_3^2})W_{2,2} - \eta W_{3,1} \\
&\tilde{\psi}_1 = W_1 + W_2
\end{aligned} \tag{10}$$

The parameters α_1^2, α_2^2 and α_3^2 can be calculated by Eq. 11:

$$\begin{aligned}
&\alpha_1^2, \alpha_2^2 = \frac{\beta^2}{2}\left(\begin{array}{c} \frac{\delta^2}{12}\left(\frac{2}{K^2(1-\upsilon)} + 1\right) \\ \pm\sqrt{(\frac{\delta^2}{12})^2\left(\frac{2}{K^2(1-\upsilon)} - 1\right)^2 + \frac{4}{\beta^2}} \end{array} \right) \\
&\alpha_3^2 = \frac{12K^2}{\delta^2}\left(\frac{\beta^2\delta^4}{72K^2(1-\upsilon)} - 1\right)
\end{aligned} \tag{11}$$

The governing equations of motion can be writing as Eq. 12:

$$\begin{aligned}
&W_{1,11} + \eta^2 W_{1,22} = -\alpha_1^2 W_1 \\
&W_{2,11} + \eta^2 W_{2,22} = -\alpha_2^2 W_2 \\
&W_{3,11} + \eta^2 W_{3,22} = -\alpha_3^2 W_3
\end{aligned} \tag{12}$$

One set of the solutions for Eq. 12 can be Eq. 13:

$$\begin{aligned}
W_1 = &[A_1\sin(\lambda_1 X_2) + A_2\cos(\lambda_1 X_2)]\sin(\mu_1 X_1) + \\
&[B_1\sin(\lambda_1 X_2) + B_2\cos(\lambda_1 X_2)]\cos(\mu_1 X_1) \\
W_2 = &[A_3\sinh(\lambda_2 X_2) + A_4\cosh(\lambda_2 X_2)]\sin(\mu_2 X_1) + \\
&[B_3\sinh(\lambda_2 X_2) + B_4\cosh(\lambda_2 X_2)]\cos(\mu_2 X_1) \\
W_3 = &[A_5\sinh(\lambda_3 X_2) + A_6\cosh(\lambda_3 X_2)]\cos(\mu_3 X_1) + \\
&[B_5\sinh(\lambda_3 X_2) + B_6\cosh(\lambda_3 X_2)]\sin(\mu_3 X_1)
\end{aligned} \tag{13}$$

In these equations, A and B are constants. λ and μ can be found by Eq. 14:

$$\begin{aligned}
&\alpha_1^2 = \mu_1^2 + \eta^2\lambda_1^2 \\
&\alpha_2^2 = \mu_2^2 - \eta^2\lambda_2^2 \\
&\alpha_3^2 = \mu_3^2 + \eta^2\lambda_3^2
\end{aligned} \tag{14}$$

It is obvious that for a simply supported edge, free edge and clamed edge Eq. 15-17 can be respectively written as:

$$\tilde{M}_{11} = \tilde{\psi}_2 = \tilde{\psi}_3 = 0 \tag{15}$$

$$\tilde{M}_{11} = \tilde{M}_{12} = \tilde{Q}_1 = 0 \tag{16}$$

$$\tilde{\psi}_1 = \tilde{\psi}_2 = \tilde{\psi}_3 = 0 \qquad (17)$$

S-C-S-F boundary condition: This boundary condition is the most complicated case and covers all possible boundary conditions. For this case, equation 18 can be written:

$$\begin{aligned}
&\lambda_1\lambda_2\lambda_3\eta^2[C_2^2L_1L_2 - C_1^2L_3L_4 - 2(C_1 - C_2)^2\mu^4(1-\upsilon)]*\\
&\cos\lambda_1 \cosh\lambda_2 \cosh\lambda_3 + (C_1 - C_2)C_2\lambda_1\mu^2\{[L_1L_2 -\\
&2(1-\upsilon)\lambda_2^2\lambda_3^2\eta^4]\sinh\lambda_2 \sinh\lambda_3 + \lambda_2\lambda_3\eta^2[L_1(1-\upsilon) - 2L_2\\
&]\}\cos\lambda_1 + (C_1 - C_2)C_1\lambda_2\mu^2\{[L_3L_4 - 2(1-\upsilon)\lambda_1^2\lambda_3^2\eta^4\\
&]\sinh\lambda_1 \sinh\lambda_3 - \lambda_1\lambda_3\eta^2[L_3(1-\upsilon) + 2L_4]\}\cosh\lambda_2 +\\
&C_1C_2\lambda_3\eta^2\{[L_1L_2\lambda_1^2 + L_3L_4\lambda_2^2\\
&]\sinh\lambda_1 \sinh\lambda_2 + \lambda_1\lambda_2[L_1L_4 - L_2L_3]\cosh\lambda_3 = 0
\end{aligned} \qquad (18)$$

In this equation, $\mu = \pi m$, $C_1 = 1 - \frac{2\alpha_2^2}{(1-\upsilon)\alpha_3^2}$ and $C_2 = 1 - \frac{2\alpha_1^2}{(1-\upsilon)\alpha_3^2}$. λ_1, λ_2 and λ_3 are functions of non-dimensional frequency parameters as Eq. 19:

$$\begin{aligned}
\lambda_1 &= \frac{1}{\eta}\sqrt{\alpha_1^2 - m^2\pi^2}\\
\lambda_2 &= \frac{1}{\eta}\sqrt{-\alpha_2^2 - m^2\pi^2}\\
\lambda_3 &= \frac{1}{\eta}\sqrt{-\alpha_3^2 + m^2\pi^2}
\end{aligned} \qquad (19)$$

RESULTS

In this part, numerical calculations of the above equations are given to clarify the method. Poisson's ratio is assumed to be equal to 0.3.

The results have high accuracy and can be used for determining the accuracy of approximate methods. To illustrate the results, a typical 3D deformed mode shapes together with their corresponding deflection counter plots for plate with aspect ratio η = 2 and thickness ratio $\delta = 0.1$ are given in Fig 2.

For different thickness to length ratios of $\delta = 0.01, 0.05, 0.1, 0.115, 0.2$ and aspect ratios of $\eta = 0.4, 0.5, 2/3, 1, 1.5, 2, 2.5$, the results are tabulated in Table 1. In Table 1, for every δ and η, the nine lowest values of frequency are displayed in ascending order.

Table 1: First nine frequencies for rectangular thick plates with boundary condition S-C-S-F

η	δ	1	2	3	4	5	6	7	8	9
0.4	0.01	10.1848	13.5947	20.0776	29.5868	39.6021	42.1851	42.9565	49.5127	57.9108
	0.05	10.1319	13.4887	19.8437	29.1124	38.9107	41.2893	42.1449	48.4091	56.3209
	0.10	9.9871	13.2121	19.2456	27.8944	37.0765	38.9971	39.9820	45.5522	52.3237
	0.15	9.7676	12.8087	18.4063	26.2480	34.6080	36.0362	37.1334	41.9122	47.4327
	0.20	9.4910	12.3200	17.4363	24.4379	31.9525	32.9523	34.1210	38.1799	42.6163
0.5	0.01	10.4206	15.7393	25.7574	39.7874	40.5324	45.0551	55.2982	60.1784	70.3559
	0.05	10.3618	15.5842	25.3653	39.0904	39.6621	44.1381	53.8756	58.4177	68.0502
	0.10	10.2054	15.1956	24.3830	37.2242	37.4665	41.7409	50.3101	54.0543	62.5113
	0.15	9.9712	14.6443	23.0502	34.6374	34.7311	38.6310	45.9063	48.7890	56.0469
	0.20	9.6782	13.9934	21.5678	31.6896	32.0545	35.3839	41.5112	43.6674	49.9152
2/3	0.01	10.9682	20.3073	37.8901	40.2293	49.6579	64.0609	67.7516	89.1850	94.2461
	0.05	10.8951	20.0257	37.0603	39.5018	48.4841	61.9924	65.5195	85.8902	90.0643
	0.10	10.7099	19.3498	35.0192	37.5793	46.5302	56.9754	60.2293	77.9793	80.6773
	0.15	10.4390	18.4298	32.4007	35.0275	41.8180	51.0561	54.0632	69.0335	70.5727
	0.20	10.1060	17.3888	29.6707	32.3003	38.0442	45.4110	48.2070	61.6093	63.2764
1	0.01	12.6728	32.9925	41.6472	62.8595	72.2171	90.4194	102.7904	111.5689	130.9964
	0.05	12.5482	32.2370	40.8218	60.7824	69.4393	86.9701	97.5322	106.1105	123.0672
	0.10	12.2606	30.4743	38.7128	55.9736	62.9527	78.8120	86.2713	94.0906	106.1656
	0.15	11.8620	28.2362	35.9677	50.3782	55.6218	69.6629	74.6338	81.5621	89.6189
	0.20	11.3931	25.8975	33.0747	45.0445	48.8911	61.3014	64.6148	70.7202	76.0573
1.5	0.01	16.7875	45.2148	60.8312	91.9180	93.5911	141.1267	149.0012	161.6466	180.1544
	0.05	16.5179	44.1176	58.4647	87.1780	89.7406	131.7987	138.3211	151.2366	164.9217
	0.10	15.9404	41.4965	53.0869	77.3057	80.9273	113.3509	116.7400	129.8348	136.6792
	0.15	15.1913	38.2377	46.9606	67.0994	71.2484	95.8630	96.8123	109.4262	112.0940
	0.20	14.3646	34.9199	41.2635	58.2641	62.5203	81.1835	81.6739	92.9209	93.3381
2	0.01	22.7512	50.6057	98.4649	99.3823	131.4853	166.1173	181.7921	250.1932	253.2971
	0.05	22.2469	49.0424	93.5256	93.9506	121.8970	154.8514	165.8679	223.8942	227.2697
	0.10	21.1870	45.5725	81.0357	84.0786	103.5900	132.2990	137.5355	178.6892	180.1780
	0.15	19.8705	41.5047	68.4310	73.5755	86.6229	111.1472	113.1589	141.1201	145.1981
	0.20	18.4903	37.5487	57.8767	64.2947	73.0862	94.1986	94.5611	114.5430	119.7053
2.5	0.01	30.5270	57.8545	105.1242	148.7010	172.2784	181.4601	232.9375	259.0458	302.2416
	0.05	29.6813	55.5986	99.6242	136.3896	159.7670	163.9473	206.9864	233.6515	263.6216
	0.10	27.8981	50.9077	88.2403	112.4394	133.2557	135.5890	165.0990	189.4319	205.4196
	0.15	25.7584	45.7332	76.6139	91.0054	107.5361	113.4230	132.1264	153.2002	162.1838
	0.20	23.6105	40.9415	66.6065	74.5669	88.3290	95.8842	108.2063	126.5933	131.8118

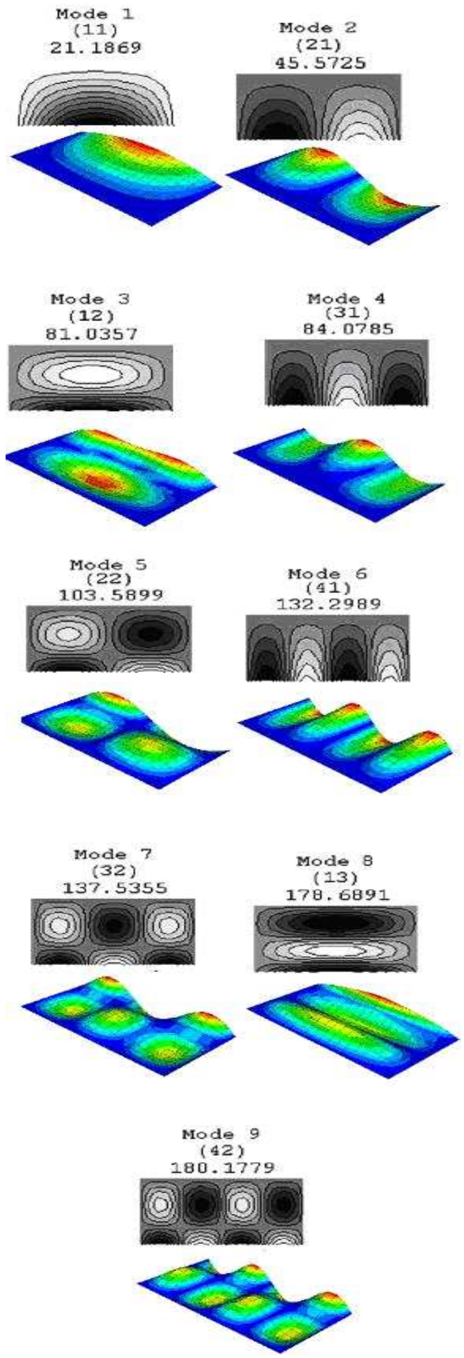

Fig. 2: First nine mode shapes of S-C-S-F rectangular plate (η = 2, δ = 0.1)

DISCUSSION

As it was mentioned before, the method which is used in this study is accurate and is based on the exact characteristic equations and no estimation is involved.

Table 2: Comparison study of frequency parameters for square mindlin plates with S-C-S-F boundaries

		Mode sequences			
δ	Method	1	2	3	4
0.1	Liew *et al.*[4]	12.2492	30.4083	38.6346	55.8018
	Present	12.2492	30.4086	38.6342	55.8017
0.2	Liew *et al.*[4]	11.3619	25.7547	32.8934	44.7241
	Present	11.3619	25.7545	32.8937	44.7244

$*K^2 = 5/6$

To assure the performance of this method, the results are compared to those of an approximate method which has acceptable accuracy Liew *et al.*[4] in the case of a rectangular plate with $\delta = 0.001$. This comparison is tabulated in Table 2 for the first four frequencies. As it can be seen, the results are close which confirm the performance of the exact method. The minor differences is because of the approximations exist in the Liew non-exact method.

CONCLUSION

In this study, Mindlin plate Theory is used to investigate the free vibration of thick rectangular plates. The general characteristic equations and transversal deformations, frequencies and different mode shapes are presented for S-C-S-F boundary condition which covers all other boundary conditions. Considering the high applicability of rectangular steel thick plates and the exact results of this method, the method can be used by engineers who need the exact results for optimize plate design.

REFERENCES

1. Dawe, D.J. *et al*., 1985. Aspects of the Analysis of Plate Structures. 2nd Edn., Clardendon Press, Oxford, ISBN: 0198561687, pp: 33.
2. Gorman, D.J., 1997. Free vibration analysis of Mindlin plates with uniform elastic edge support by the superposition method. J. Sound Vibrat., 207: 335-350. DOI: 10.1006/jsvi.1997.1107
3. Hashemi, S.H. and M. Arsanjani, 2004. Exact characteristic equations for some of classical boundary conditions of vibrating moderately thick rectangular plates. Int. J. Solids Struct., 42: 819-853. DOI: 10.1016/j.ijsolstr.2004.06.063
4. Liew, K.M. and T.M. Teo, 1999. Three-dimensional vibration analysis of rectangular plates based on differential quadrature method. J. Sound Vibrat., 20: 577-599. DOI: 10.1006/jsvi.1998.1927