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Resonance Caused by the Gravitational waves On an Earth Satellite

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Abstract: The present work deals with the motion of an Earth satellite taking into account the oblateness of the Earth and of a passing Gravitational wave. The oblateness of the Earth is truncated beyond the second zonal harmonic, J_2, which plays the role of the small parameter of the problem. The conditions for resonance are determined and the resonance resulting from the commensurabilities between the wave frequency and the mean motions of the satellite, the nodal regression, and the apsidal rotation are analyzed.

Key words: Gravitational waves, earth-satellite, earth motion, resonance problem.

INTRODUCTION

Resonance problems occur frequently in nonlinear mechanics and celestial mechanics . It is usually manifested by the appearance, when integrating the equations of motion, of small divisors of the form $D = (K.n) = K_1 n_1 + ... + K_m n_m$, where the components of the resonant vector K are integers and the n's are the fundamental frequencies of the system. If there exits one such resonant vector (i.e., one small divisor) the resonance is called simple, otherwise it is multiple. Due to the importance of these commensurate orbits, they have received much attention.

In recent years, the general theory of relativity became a necessary framework for the construction of accurate dynamical ephemerides and in the discussion of high precision observations. Regarding general relativity, the structure of the field equations and the equations of motion is the subject of relativistic celestial mechanics.

An important consequence of general relativity is the existence of gravitational waves produced by changes in the distribution of matter not symmetrical about a point. These waves travel in all space with the velocity of light. Recently efforts has been directed toward detecting them by the dynamical effects that they may produce in heavenly bodies, this effect results in increasing the distance between two particles by about 10^{-17} the natural separation.

An estimation was provided for the amplitude, duration and the frequency of arrival at earth of gravitational wave bursts expected from the activity of the nuclei of distant galaxies and quasars [1]. It is estimated that they might reach at earth as often as 50 times per year, or as rarely as once each 300 years. Also, it is suggested that such bursts may be detected using dual-frequency Doppler tracking of interplanetary spacecrafts.

Cylindrical coordinates was used to find the first order orbital variations. It is found that for an initially circular orbit, resonance occurs for γ = 1,2,3 and that for elliptic orbits the resonance occurs for any positive integral value of γ depending on the approximation scheme [2].

Lagrange's planetary equations was used to find a first order solution in all the elements in the case of oblique incidence of the wave [3] .

Fundamental models are the simplest, one degree of freedom Hamiltonians that serve as a tool to understand the qualitative effects of various resonances. A new extended fundamental model was proposed in order to improve the classical, Andoyer type, second fundamental model [4].

The present work treats the resonance arising from the commensurability between the wave frequency and the mean motions of the satellite, the nodal regression, and the apsidal rotation.

THE ACCELERATION COMPONENTS

To find the acceleration components produced by the waves on a bound system of two bodies (e.g. an Earth-Satellite, or a planetary system) we assume that the characteristic dimension of the system is small compared to the length of the wave, and the velocities of each component of the system are much smaller than the speed of light, so that our frame is locally inertial. To do so we now proceed to find the acceleration components produced by the wave at a point x, y, z. Let us start with the equation of geodesic deviation [5]

$$\frac{d^2\eta^\gamma}{ds^2} + R^\gamma{}_{smn} p^s \eta^m p^n = 0 \quad (1)$$

The field of a weak gravitational wave is determined by a metric close to the Minkowski [3]

$$\begin{aligned} ds^2 &= c^2dt^2 - (1-h_{11})dx^2 - (1+h_{11})dy^2 + 2h_{12}dxdy \\ 1 &= c^2\left(p^0\right)^2 - (1-h_{11})\left(p^1\right)^2 - (1+h_{11})\left(p^2\right)^2 + 2h_{12}\left(p^1\right)\left(p^2\right) \end{aligned},$$

but

$$\frac{1}{c}p^l = \frac{d\eta^l}{dt}\frac{dt}{ds} \to 0$$

so that for $c = 1$, $ds = dt$ and $p^5 = (1,0,0,0)$, Eq. (1) reduces to

$$\frac{d^2\eta^\gamma}{ds^2} + R^\gamma{}_{omo}\eta^m = 0\,, \quad (2)$$

or simply

$$\frac{d^2\eta_i}{dt^2} + R^\gamma{}_{iomo}\eta^m = 0 \quad (3)$$

now retaining orders up to order h_{ik}, we have

$$R_{iklm} = \frac{1}{2}\left[\frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{km}}{\partial x^i \partial x^l}\right]$$

From which ($h_{io} = h_{io} = 0$) $R_{i0l0} = -\frac{1}{2}\frac{\partial^2 h_{il}}{\partial t^2}$ and Eq. (3) yields

$$\frac{d^2x}{dt^2} = \frac{1}{2}\frac{\partial^2 h_{1m}}{\partial t^2}\eta^m\,,$$

$$\frac{d^2y}{dt^2} = \frac{1}{2}\frac{\partial^2 h_{2m}}{\partial t^2}\eta^m\,,$$

$$\frac{d^2z}{dt^2} = \frac{1}{2}\frac{\partial^2 h_{3m}}{\partial t^2}\eta^m\,.$$

remembering that for a plane wave traveling in the Z-direction the only non-vanishing components are

$$h_{11} = -h_{22}\,,\ h_{12} = h_{21} \quad (4)$$

so that we have

$$\frac{d^2x}{dt^2} = \frac{1}{2}\frac{\partial^2 h_{11}}{\partial t^2}x + \frac{1}{2}\frac{\partial^2 h_{12}}{\partial t^2}y \quad (5)$$

$$\frac{d^2y}{dt^2} = \frac{1}{2}\frac{\partial^2 h_{12}}{\partial t^2}x - \frac{1}{2}\frac{\partial^2 h_{11}}{\partial t^2}y \quad (6)$$

$$\frac{d^2z}{dt^2} = 0\,. \quad (7)$$

Now Eq. (4) shows that the wave is transverse with two states of polarization. Remembering that the h_{ij} satisfy the wave Eq.

$$\Upsilon h_{ij} = 0$$

we choose

$$h_{11} = h_x \cos(n_\omega t + \alpha_1)\,,$$

$$h_{12} = h_y \cos(n_\omega t + \alpha_2)$$

Where h_x, h_y are the dimensionless amplitudes of the wave in two orthogonal directions in the transverse plane, α_1 and α_2 are the phase differences. Υ is the D'Alembertian operator.

THE HAMILTONIAN

The Hamiltonian of the problem may be written in the form

$$H = H_E + H_w \quad (8)$$

where H_E represents the contribution of the earth's shape, while H_W represents the contribution of the gravitational waves. Utilizing the acceleration components of the gravitational waves given by equations (5)-(7), the Hamiltonian, H_W, may be written in the form

$$H_w = \frac{1}{2}h_1\left(y^2 - x^2\right) - h_2xy \qquad (9)$$

where $h_1 = \frac{1}{2}\mu\frac{\partial^2 h_{11}}{\partial t^2}$ and $h_2 = \frac{1}{2}\mu\frac{\partial^2 h_{12}}{\partial t^2}$. μ is the reduced mass of the system, and x,y are given by [6]

$$x = \left[\cos l_3 \cos\left(f + l_2\right) - \cos i \sin l_3 \sin\left(f + l_2\right)\right]$$

$$y = \left[\sin l_3 \cos\left(f + l_2\right) + \cos i \cos l_3 \sin\left(f + l_2\right)\right]$$

In terms of the Delaunay set of canonical elements, Noting that the time t appears explicitly in the expression (9) for the Hamiltonian through the term $n_w t$, the Hamiltonian may be written as [7]

$$H_w = n_w L_4 + \sum_{k\geq o}\sum_{m=-2}^{2}\sum_{i=-2}^{2}\sum_{j=-1}^{1} L^4{}_1 \begin{bmatrix} B^{ij}{}_{km}\cos\left(kl_1 + ml_2 + il_3 + jl_4\right) + \\ +\bar{B}^{ij}_{km}\sin\left(kl_1 + ml_2 + il_3 + jl_4\right) \end{bmatrix} \qquad (10)$$

where ($m = 0,\pm 2$, $i = \pm 2$, $j = \pm 1$), n_w is the frequency of the wave, $\varepsilon = \frac{1}{2}\mu n_w{}^2 h_x$, L_4 the conjugate of l_4, , and the Delaunay variables are augmented by the pair (l_4, L_4). The coefficients $B_{km}{}^{ij}$, $\bar{B}_{km}{}^{ij}$ are all cited in reference [7].

Our set of canonical elements now consists of

$l_1 = l$ = mean anomaly , $L_1 = \sqrt{\mu a}$

$l_2 = g = \omega$ = argument of perigee , $L_2 = L_1\sqrt{1 - e^2}$

$l_3 = h$ = longitude of the node , $L_3 = L_2 \cos I$

$l_4 = n_w t$, L_4

or $l_i \equiv \left(l, g, h, l_4\right)$, $L_i \equiv \left(L, G, H, L_4\right)a$, $\left(i = 1,2,3\right)$.

The Hamiltonian of the problem, up to the second order, can now be expressed as a power series in J_2, as follows

$$H_o = \frac{\mu^2}{2L_1{}^2} + n_w L_4 \qquad (11)$$

$$H_1 = \frac{\mu^4 R^2}{L_1{}^6}\left(\frac{a}{r}\right)^3 \begin{bmatrix} \frac{1}{4} - \frac{3L_3{}^2}{4L_2{}^2} - \frac{3}{4} \\ \left(l - \frac{L_3{}^2}{L_2{}^2}\right)\cos 2\left(l_1 + l_2\right) \end{bmatrix}, \qquad (12)$$

$$H_2 = \sum_{k\geq o}\sum_{m=-2}^{2}\sum_{i=-2}^{2}\sum_{j=-1}^{1} L^4{}_1 \begin{bmatrix} B^{ij}{}_{km}\cos\left(kl_1 + ml_2 + il_3 + jl_4\right) + \\ +\bar{B}^{ij}_{km}\sin\left(kl_1 + ml_2 + il_3 + jl_4\right) \end{bmatrix} \qquad (13)$$

where H_1 represents the contribution of the earth's oblateness, up to order J_2, to the Hamiltonian.

SHORT PERIOD ELIMINATION

The Hamiltonian of the problem, H^*, after eliminating the short period element, using a perturbation technique based on Lie series and Lie transform[8], can be represented as (in the following all the variables are understood to be single primed, but the primes are dropped for the sake of simplicity of writing).

$$H^* = H_o{}^* + H_1{}^* + H_2{}^* \qquad (14)$$

where

$$H_o{}^* = -\frac{\mu^2}{2L_1{}^2} + n_w L_4, \qquad (15)$$

$$H_1{}^* = \frac{\mu^4 R^2}{4L_1{}^3 L_2{}^3}\left(3\sin^2 I - 2\right), \qquad (16)$$

$$H_2{}^* = \sum_{m=-2}^{2}\sum_{i=-2}^{2}\sum_{j=-1}^{1} L_1{}^4 B^{iij}{}_{o\upsilon}\cos\left(ml_2 + il_3 + jl_4\right). \qquad (17)$$

Noting that the terms ($ml_2+il_3+jl_4$) appear in the equation for H^* with the result that resonant terms may arise, these will affect the ordering process of transformation.

TRANSFORMATION IN THE CASE OF RESONANCE

Now the system of canonical equations of motion is:

$$\dot{l}_i = \frac{\partial H^*}{\partial L_i},\ \dot{L}_i = -\frac{\partial H^*}{\partial l_i},\ (i = 1,.......,4) \tag{18}$$

The system (18) has 3-degrees of freedom (remembering that l_i, L_i are single primed). We perform a canonical transformation in which a new angle variable is introduced replacing one of the fast variables producing the resonant vector. The introduced variable (known as the Delaunay anomaly [9]) is chosen such that it becomes a slow variable.

Let us perform the transformation $l_i', L_i' \to y_i, x_i, H^* \to \bar{H}$ that reduces the system (18) to be of one degree of freedom.

Let $y_1 = l l_1'$, $y_2 = l_2'$, $y_3 = l_3'$ and $y_4 = ml_2' + il_3' + jl_4'$.

this leads to the relations

$$x_1 = L_1',\ x_2 = L_2' - \frac{m}{j}L_4',$$
$$x_3 = L_3' - \frac{i}{j}L_4',\ x_4 = \frac{L_4'}{j},$$

and the system (18) becomes

$$\dot{y}_i = \frac{\partial \bar{H}}{\partial x_i},\ \dot{x}_i = -\frac{\partial H}{\partial y_i},\ (i = 1,.......,4), \tag{19}$$

Where

$$\bar{H}_o = -\frac{\mu^2}{2x_1^2} + jn_w x_4, \tag{20}$$

$$\bar{H}_1 = \frac{\mu^4 R^2}{4x_1^3 (x_2 + mx_4)^3}\left(3\sin^2 I - 2\right), \tag{21}$$

$$\bar{H}_2 = \sum_{m=-2}^{2}\sum_{i=-2}^{2}\sum_{j=-1}^{1} x_1^4 B_{om}^{ij}\left(1 - 2\sin^2\frac{y_4}{2}\right). \tag{22}$$

Equation (19) represent the transformed system, and it is now of one degree of freedom.

THE CONDITIONS FOR RESONANCE

Carrying out the procedure to determine the second order generator W_2^*, using the Lie series technique, a resonant vectors of the form $\left(m\dot{l}_2 + i\dot{l}_3 + j\dot{l}_4\right)$ will arise in the denominators of W_2^* When none of the combinations $\left(m\frac{\partial H_1^*}{\partial L_2} + i\frac{\partial H_1^*}{\partial L_3} + jn_w\right)$ vanishes or remains of the first order, this represents the non resonance case. If some of the denominators vanish or reduce to the second order, the equation for W_2^* looses its validity (resonance case).

Let us define the following mean motions:

$$n = \frac{\mu^2}{L_1^2}, \tag{23}$$

$$n_2 = \frac{\partial H_1^*}{\partial L_2} = nJ_2\eta^{-4}\left(\frac{r_e}{a}\right)^2\left(\frac{15\cos I}{4} - \frac{3}{4}\right), \tag{24}$$

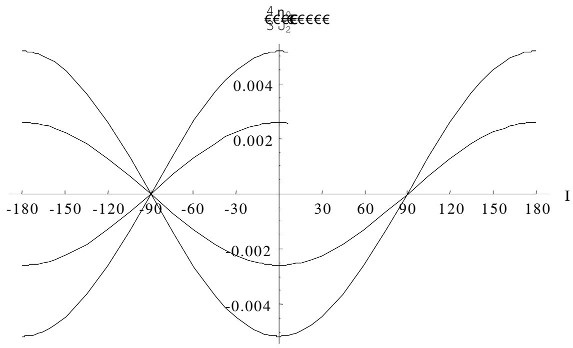
$$n_3 = \frac{\partial H_1^*}{\partial L_3} = nJ_2\eta^{-4}\left(\frac{r_e}{a}\right)^2\left(-\frac{3\cos I}{2}\right), \tag{25}$$

$$n_4 = n\upsilon_o\left(\frac{r_e}{a}\right)^{\frac{3}{2}}. \tag{26}$$

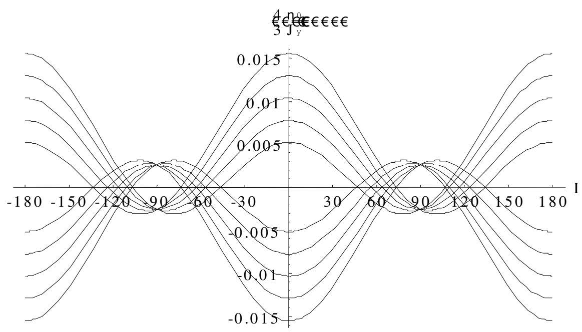
where $\eta = \frac{L_1}{L_2}$, and V_0 is the ratio of the frequency of the wave, n_w, to the mean motion of a fictitious satellite with r_e as its semi-major axis [10]. Using equations (23)-(26), yields

$$mn_2 + in_3 + jn_4 = nJ_2\eta^{-4}\left(\frac{r_e}{a}\right)^2\left[\left(\frac{15}{4}m\cos^2 I - i\frac{3}{2}\cos I - \frac{3}{4}m\right) - j\frac{\upsilon_o}{J_2}\left(\frac{a}{r_e}\right)^{\frac{7}{2}}\eta^4\right]$$

hence the conditions for resonance is given by



a



b

Fig. 1: (a) case m = 0, (b) Case m = 2.

$$j\left(\frac{15}{4}m\cos^2 I - i\frac{3}{2}\cos I - \frac{3}{4}m\right) = \frac{3\upsilon_o}{4J_2}\left(\frac{a}{r_e}\right)^{\frac{7}{2}}\eta^4 . \qquad (27)$$

where ($m = 0,1,2$; $i = -2,-1,0$; $j = 0,\pm 1$). The conditions (27) is a relation between a, e and cosI that yield a number of curves (corresponding to the different values of m shown in the next Fig.

CONCLUSION

The canonical equations of motion of the problem under concern are formulated including the effects of the earth's oblateness and of a passing gravitational wave. The conditions of the resonance are determined and written in a general form. As a numerical example, we adopted the Molniya-type satellites (highly eccentric Earth orbits, e = 0.75) to reveal the different types of resonance. A case of simple resonance is obtained when a set of values a, e and cosI is such that it lies on any of the curves arcs. When more than one denominator tends to zero (or $O(J_2^2)$), we have a case of multiple resonance. In the figures this occurs when a set of values of a, e and cosI fits across of any two or more curves.

The Fig. 1a and b show the known resonance cases, such as the critical inclination, (b) case (I = 63.3°, I = 116.6°) and the polar orbits case (I = 90°).

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